Improving Longevity and Mortality Risk Models using Common Stochastic Long-Run Trends

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Abstract

It is well-known that dependancies exist between causes of death, as well as across age-groups. Over the last century, the assumption has been usually made that causes of death, or age-groups, are independent when they are considered. Recent developments in econometrics allow, through Vector Error Correction Models (VECM), to model multivariate dynamic systems including time dependency between economic variables. In this paper, VECM are developed for causes of death and age-based risk factors.

We analyze the five main causes of death across nine major countries. Our analysis shows that long-run equilibrium relationships exist between the five main causes of death, improving our understanding of the dependence between these competing risks. Similarities between the countries under study are observed, with groups of countries having similar experience.

Furthermore, projecting mortality within a country must take into account dependencies across age-groups. In order to do so, a parameterized mortality model, the Heligman-Pollard function, is used in place of the highly parameterized Lee-Carter model. This is estimated using data over a period of 50 years. The parameters are modeled as stochastic factors using a VECM, taking into account time dependency and long-run trends between the parameters not allowed for using more traditional ARIMA processes. The VECM approach is a significant improvement over ARIMA models allowing a more realistic quantification of risk including parameter risk.

1 Introduction

Longevity and mortality improvement has been driven by many factors over the last century. There have been systematic changes in causes of death and these have been common across the developing economies. At the same time, the factors have impacted different ages resulting in dependence in improvements across age-groups. As a result, longevity and mortality risk across countries and within a country across ages contains common stochastic trends. It is important to incorporate these common trends in longevity and mortality risk models used for risk management.

Vector Autoregressions as well as Vector Error Correction Models (VECM) have been developed in econometrics to model multivariate dynamic systems including time dependency between economic variables. They incorporate common stochastic trends that exist between the variables, along with long-run equilibrium relationships. In other words, some variables may move together over time and thus are linked to each other by some relations. VECM are tools that can be used in order to catch these interactions. In this paper, VECM are developed for causes of death and age-based risk factors.

Causes of death are believed by many demographers and other experts to be key factors needed in mortality modeling (see for example Gutterman and Vanderhoof [1998] or Tabeau et al. [1999]). Indeed, causes of death reflect underlying socio-economic factors and provide important insights into longevity and mortality modeling. Despite this, mortality has not generally been analyzed by cause of death from the perspective of modeling future trends. Cause of death analysis has limitations due to cause of death reporting being less reliable at older ages where most of the deaths occur. Indeed, inaccuracy of reported age at death may occur as well as sampling errors of estimated mortality rates when the number of deaths is small (Tuljapurkar and Boe [1998]). Multiple causes can be present. Misclassifications of deaths by cause occur as well. Differences in interpretation of international rules, in coding practices and in training of physicians may also differ across countries (Myers [1996]). Because of data limitations for disaggregated data, time series analysis and model parameters for causes of death are often less stable than parameters in an aggregate mortality model (Wilmoth [1995]). Finally, causes of death represent competing risks which involves a dependence that should be taken into account in the modeling process.

It remains unknown how to insert in a model these interrelations between causes. Over the last decades, to overcome this issue, an independence assumption has been used in many theoretical models. Cause elimination models as well as cause-delay models developed by Manton et al. [1980] and Jay Olshansky [1987] are two well-known examples. Tabeau et al. [1999] as well as McNown and Rogers [1992] have considered the impact on projections of modeling mortality rates by cause of death, assuming independent causes. However, our analysis shows that long-run equilibrium relationships exist between the five main causes of death. It improves our understanding of the dependence between these competing risks and may help, in further researches, in building new forecasting models that include these relationships.

Projecting mortality must also take into account dependencies across age-groups. Empirical evidence shows that multiple factors have driven changes in mortality rates and that these factors have impacted different ages to varying extents. In order to capture this age dependence of cause-specific mortality rates, we use an approach similar to the one of McNown and Rogers [1992]: The age pattern of cause-specific mortality rates is represented by a parameterized mortality model, the Heligman-Pollard function. The parameters have interpretation as factors impacting specific age ranges. This function is estimated over the period 1950–2000. The resulting time series of the parameters are forecasted using a VECM, allowing to take into account time dependency and long-run trends between the parameters of the function, in contrast to the more traditional AutoRegressive Integrated Moving Average (ARIMA) processes used by McNown and Rogers [1992]. The VECM approach proves to be a significant improvement over ARIMA models allowing a more realistic quantification of risk including parameter risk.

The following paper starts with an introduction on the theoretical backgrounds needed for a VECM analysis (Section 2). Long-run equilibriums between age-based risk factors are then developed in Section 4.1, employing the data set described in Section 3. It leads to a new mortality forecasting approach introduced in Section 4.2. Finally, VECM are developed for the five main causes of death in nine major countries (Section 4.3).

2 VAR and VECM, Theoretical Background

Vector AutoRegressions (VAR) describe the dynamic interactions existing between a set of variables, such as the price of a good in Australia, the price of the same good in China and the exchange rate between the two currencies. This multivariate time series approach became popular among the economists a few decades ago and is often used to model time series of economic variables.

Extensions of this model, such as the Vector Error Correction Model (VECM) or cointegration modeling, provide useful information on the relations binding the variables under study. By interpreting cause-specific mortality rates as different variables, a VAR model may be fitted and interactions between causes analyzed.

A pth-order vector autoregression, denoted a VAR(p), explains each variable with its own p lags as well as with the p lags of the other variables considered in the model, that is

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \epsilon_t. \tag{1}$$

The n variables under study at time t are collected in the $(n \times 1)$ vector $\mathbf{y}_t$, while $\mathbf{c}$ is an $(n \times 1)$ vector of constants and $\mathbf{\Phi}_i$ is an $(n \times n)$ matrix of autoregressive coefficients for $i = 1, 2, \ldots, p$. The $(n \times 1)$ vector ϵ_t is a vector of white noises, such that

$$E(\epsilon_t) = \mathbf{0}, \tag{2}$$

$$E(\epsilon_t \epsilon_l) = \begin{cases} \mathbf{\Omega} & \text{for } t = l \\ \mathbf{0} & \text{for } t \neq l, \end{cases} \tag{3}$$

where $\mathbf{\Omega}$ is a symmetric positive definite matrix.

Estimates of the parameters of a VAR(p) and the associated asymptotic distributions rely on the stationarity of the process. A VAR(p) is called stationary if its first and second moments are constant over time, that is $E(\mathbf{y}_t)$ and $E(\mathbf{y}_t \mathbf{y}'_{t-j})$ are independent of the date t, but may depend on the time difference j. It implies that the process has a constant mean (no trend) and its variance does not change over time. However, many economic time series do have a trend. Stationary processes are too restrictive in order to be able to capture all the features existing in the observed time series. Non-stationarity should then be considered as well.

A non-stationary variable (x_t) often becomes stationary once its first difference is computed, that is

$$\nabla x_t = x_t - x_{t-1}.$$

This variable is called integrated of order one, denoted $I(1)$. Thus, an idea could be to work on the first difference of the data under observation. If the process is integrated of order one, that would remove the non-stationarity and a VAR(p) could be fitted. However, differentiation could distort some interesting relations existing between the original variables. Cointegrated variables are an example.

As already stated above, many economic variables are non-stationary. However, it could happen that the variables move together in the same direction over time. In such a situation, they may have a common trend, which can be represented by a long-run equilibrium relationship. A linear combination of these variables would exist such that the relation is stationary. Such variables are called cointegrated.

Suppose that there are n $I(1)$ variables y_t under observation and that they are tied to one another by the relation

$$\beta_1 y_{1t} + \beta_2 y_{2t} + \cdots + \beta_n y_{nt} = 0.$$

This equilibrium relationship is valid on the long view. For a particular point in time, there might be some deviations from that equilibrium such that

$$\beta_1 y_{1t} + \beta_2 y_{2t} + \cdots + \beta_n y_{nt} = z_t, \tag{4}$$

where z_t is a stochastic variable representing that deviation (Lütkepohl [2005]). If a long-run equilibrium exists, z_t should be stationary. In such a situation, the integrated variables are cointegrated.

Equation 4 may be written with vector and matrix notations as

$$\beta' \mathbf{y}_t = z_t, \tag{5}$$

with

$$\begin{aligned} \beta &= (\beta_1 \quad \beta_2 \ldots \beta_n)', \\ \mathbf{y}_t &= (y_{1t} \quad y_{2t} \ldots y_{nt})'. \end{aligned} \tag{6}$$

The vector β is called the cointegrating vector. More than one cointegration relation may exist, and thus there might be more than one cointegrating vector, each being linearly independent from the others. For example, if there are five variables, the first two may be linked by a cointegration relation and the last three by another one. In such a situation, the vector β of equation 6 becomes a matrix with each of its column being a cointegrating vector, that is

$$\begin{aligned} \beta &= (\boldsymbol{\beta_1} \quad \boldsymbol{\beta_2} \ldots \boldsymbol{\beta_r}), \\ &= \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1r} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2r} \\ \vdots & & & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nr} \end{pmatrix}, \end{aligned} \tag{7}$$

with $\boldsymbol{\beta_i}$ the ith cointegration relation, for $i = 1, 2, \ldots, r$. The stationary vector $\beta' \mathbf{y}_t$ contains the r linearly independent cointegrated relations of the n variables under study.[1]

If the columns of β of equation 7 represent all the linearly independent relations, that is all other cointegrated relations are a linear combination of the columns of β, then it is said that there are exactly r cointegrating relations among the elements of $\mathbf{y}_t$ and that $(\boldsymbol{\beta_1} \quad \boldsymbol{\beta_2} \ldots \boldsymbol{\beta_r})$ form a basis of the space of cointegrating vectors (Hamilton [1994]).

It is worth noting that the cointegrated relations are not uniquely defined and so is the matrix β. Indeed, each cointegrating vector could be multiplied by a constant and the relationship will remain the same. Broadly

[1] In this paper, we are interested in variables integrated of order one. In that special case, cointegrated relations are necessarily stationary. For a more general framework, see Hamilton [1994] and Lütkepohl [2005].

speaking, for any non-zero vector $\mathbf{v}$ with r elements, $\mathbf{v}'\beta'\mathbf{y}_t$ also represents the r cointegrated relations. Therefore, constraints have to be imposed on some values of the matrix β.

The cointegration relations may be used in VAR modeling. A VAR(p) can equivalently be written as (see, for example, Hamilton [1994] for a proof)

$$\nabla \mathbf{y}_t = \mathbf{c} + \xi_1 \nabla \mathbf{y}_{t-1} + \xi_2 \nabla \mathbf{y}_{t-2} + \cdots + \xi_{p-1} \nabla \mathbf{y}_{t-p+1} + \mathbf{\Pi} \mathbf{y}_{t-1} + \epsilon_t, \quad (8)$$

where

$$\begin{aligned}
\mathbf{\Pi} &= -(\mathbf{I}_n - \mathbf{\Phi_1} - \cdots - \mathbf{\Phi_p}); \\
&= \alpha\beta'; \\
&= \text{matrix of rank } r; \\
\alpha &= \text{a } (n \times r) \text{ loading matrix ;} \\
\beta &= \text{a } (n \times r) \text{ matrix containing the } r \text{ vectors} \\
&\quad\ \text{forming a basis of the space of cointegration;} \\
\xi_i &= -(\mathbf{\Phi_{i+1}} + \cdots + \mathbf{\Phi_p}) \quad \text{for } i = 1, \ldots, p-1.
\end{aligned}$$

Expression 8 is known as the Vector Error Correction Model of the cointegrated system. Each element is stationary as the first difference of a $I(1)$ process is stationary as well as the cointegration relations. The loading matrix α indicates which cointegrated relation has an impact on which variable and to which extent. For example, the element α_{ij} measures the effect of the cointegrated relation j ($j = 1, \ldots, r$) on the variable i ($i = 1, \ldots, n$).

The rank of the matrix $\mathbf{\Pi}$ indicates the number of cointegrated relations among the variables of the process. Three different cases appear:

Case 1: $r = 0$ There is no cointegrated relation. A VAR($p-1$) may be applied on the first difference of the variables.

Case 2: $r = n$ All linear combinations are stationary. Thus, all the variables in the process are stationary.

Case 3: $0 < r < n$ There are r cointegrated relations, such that $\mathbf{\Pi} = \alpha\beta'$. In that case, the cointegrated relations are included in the error correction term.

The Johansen's approach can be used in order to find the number of cointegrated relations in a process as well as the value of the matrices α, β, $\mathbf{c}$ and ξ_i for $i = 1, 2, \ldots, (p-1)$ in equation 8 (Hamilton [1994] and Lütkepohl [2005]).

As a summary, in order to conduct a VECM analysis, the following steps should be taken (Figure 1):

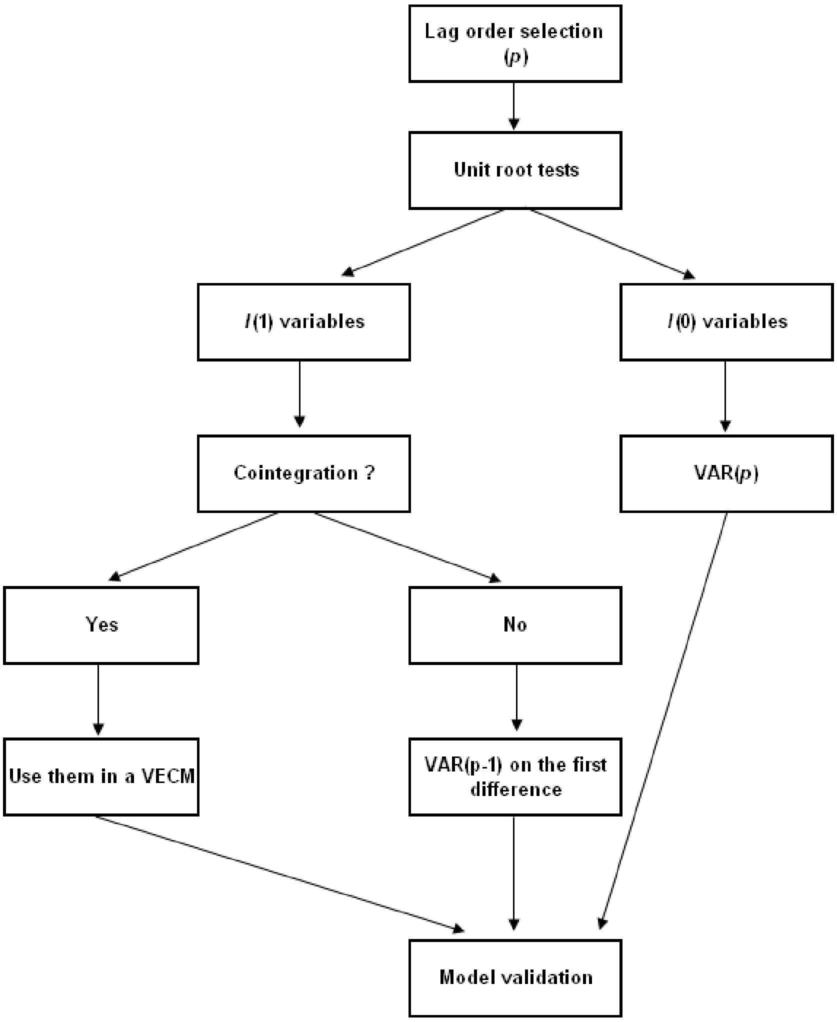


Figure 1: Steps to follow in a VECM analysis

1. Lag order of the VAR, p: Using selection criteria, such as Akaike's Information Criteria (AIC), Hannan-Quinn Criterion (HQ), Schwarz Criterion (SC), Final Prediction Error (FPE), the lag order of the VAR is selected.

2. Unit root tests on the variables considered: For a process to be stationary, the characteristic polynomial of its VAR should have all its roots outside the complex unit circle (Hamilton [1994] and Lütkepohl [2005]). Therefore, if this polynomial has a root equal to unity, some or all the variables are integrated of order one and there might be cointegrated relations among them. Unit root tests, such as the Kwiatkowski-Phillips-Schmidt-Shin test (KPSS), the Augmented Dickey-Fuller test (ADF) or the Phillips-Perron test (PP), are useful tools in order to check for the stationarity of the variables. KPSS tests the null hypothesis that the variable is level or trend stationary, while ADF and PP test the null hypothesis of a unit root, and thus, the null hypothesis of non-stationarity.

3. If the variables are stationary, denoted $I(0)$, a VAR(p) is suitable. If the variables are $I(1)$, the Johansen's procedure may be applied in order to find the number of cointegrated relations. Two test statistics are commonly used in order to find the number of cointegrated relations: the trace test and the maximum-eigenvalue test.

4. If the variables are $I(1)$ and if there is no cointegration, a VAR($p-1$) on the first difference may be computed. Otherwise, the appropriate VECM should be found.

5. Model validation: tests for residual autocorrelations and non-normality.

3 Data

The VECM analysis is applied on five-year age-group cause-specific mortality rates collected from the Mortality Database administered by the World Health Organization (WHO, World Health Organization [2009]). Nine countries are chosen from the developed world – North America, Europe, Asia and Oceania. The selected countries and causes of death are shown in Table 1 for males and Appendix A for females along with the periods covered.

Causes of death are defined by the International Classification of Diseases (ICD), which ensures consistencies between countries (Appendix B, Table 13). This classification changed three times between 1950 and 2006, from ICD-7 to ICD-10, in order to take into account progresses in science and technology and to refine the classification. The raw data are then not directly comparable for different periods. To make them comparable, comparability

Table 1: Sample years of the percentages of deaths by cause, males

Country	Cause of death	1955	1970	1985	2000
USA (1950 - 2005)	Circulatory system	42.90%	51.31%	44.40%	37.19%
	Cancer	14.73%	16.92%	22.77%	24.85%
	Respiratory system	4.36%	6.72%	8.48%	9.40%
	External causes	9.55%	10.52%	9.51%	8.89%
	I&P	2.02%	0.91%	1.34%	2.70%
	Total	73.55%	86.38%	86.51%	83.03%
Australia (1950 - 2003)	Circulatory system	39.46%	51.59%	45.07%	35.46%
	Cancer	13.85%	16.13%	24.59%	30.82%
	Respiratory system	6.67%	9.76%	8.98%	8.93%
	External causes	9.69%	9.67%	8.74%	8.33%
	I&P	2.12%	0.87%	0.48%	1.30%
	Total	71.79%	88.03%	87.85%	84.83%
Switzerland (1951 - 2005)	Circulatory system	30.10%	40.52%	42.52%	36.21%
	Cancer	19.58%	22.24%	29.48%	28.87%
	Respiratory system	5.55%	7.45%	6.25%	7.95%
	External causes	10.78%	11.36%	9.58%	7.81%
	I&P	3.65%	1.55%	0.68%	1.11%
	Total	69.66%	83.12%	88.52%	81.95%
Japan (1950 - 2006)	Circulatory system	9.33%	39.85%	35.84%	27.50%
	Cancer	11.97%	18.09%	28.02%	34.97%
	Respiratory system	6.69%	8.08%	10.48%	14.36%
	External causes	10.98%	11.23%	9.54%	9.38%
	I&P	10.22%	3.86%	1.74%	2.07%
	Total	49.19%	81.10%	85.62%	88.28%
Singapore (1963 - 2006)	Circulatory system	NA	26.64%	33.05%	34.76%
	Cancer	NA	15.51%	22.93%	29.42%
	Respiratory system	NA	13.83%	17.62%	16.87%
	External causes	NA	10.11%	10.37%	7.50%
	I&P	NA	8.06%	3.05%	1.72%
	Total	0.00%	74.14%	87.03%	90.28%

Table 1: Sample years of the percentages of deaths by cause, males - continue

Country	Cause of death	1955	1970	1985	2000
Italy (1951 - 2002)	Circulatory system	24.65%	41.02%	41.15%	38.27%
	Cancer	14.14%	20.20%	27.83%	32.73%
	Respiratory system	8.89%	10.45%	8.07%	7.81%
	External causes	7.25%	7.29%	6.34%	5.62%
	I&P	4.70%	2.14%	0.47%	0.64%
	Total	59.63%	81.09%	83.87%	85.07%
Norway (1951 - 2005)	Circulatory system	29.72%	50.43%	48.55%	39.95%
	Cancer	18.64%	18.61%	22.40%	26.04%
	Respiratory system	5.32%	8.33%	9.01%	9.47%
	External causes	8.38%	7.38%	7.11%	6.58%
	I&P	2.64%	0.79%	0.67%	1.11%
	Total	64.71%	85.56%	87.73%	83.15%
Sweden (1951 - 2005)	Circulatory system	37.01%	52.36%	54.18%	45.67%
	Cancer	17.32%	19.99%	21.09%	24.96%
	Respiratory system	5.22%	6.00%	8.43%	7.27%
	External causes	8.69%	8.47%	6.32%	5.91%
	I&P	1.80%	1.01%	0.66%	1.08%
	Total	70.04%	87.82%	90.67%	84.88%
United Kingdom (1950 - 2006)*	Circulatory system	36.40%	48.55%	48.04%	39.83%
	Cancer	18.13%	21.40%	25.19%	28.52%
	Respiratory system	12.60%	16.12%	11.73%	12.13%
	External causes	4.93%	4.44%	4.05%	4.41%
	I&P	2.42%	0.76%	0.41%	0.81%
	Total	74.49%	91.26%	89.42%	85.70%

I&P = Infectious and parasitic diseases.
The years in brackets under the country name represent the period under observation.
The table should be read as follows: In 2000, in the United States, 37.19% of the deaths were because of the diseases of the circulatory system. The five main causes of death caused 83.03% of the deaths.

* No data are available for United Kingdom in 2000. The percentages of 2001 are presented instead.

ratios are computed, where

$$CR_{x,c,d,s,j_{c,k},i_{c,k+1}} = \frac{q_{x,c,i_{c,k+1},d,s} + q_{x,c,(i_{c,k+1}+1),d,s}}{q_{x,c,j_{c,k},d,s} + q_{x,c,(j_{c,k}-1),d,s}},$$

with

$CR_{x,c,d,s,j,i}$ = comparability ratio for age x, country c, cause of death d, sex s, between year j and i;

$q_{x,c,t,d,s}$ = probability of dying at age x, in country c, and in year t, from cause of death d, for a person of sex s;

$i_{c,k}$ = first year for country c in classification k;

$j_{c,k}$ = last year for country c in classification k;

$k \in \{7, 8, 9\}$.

The aim is to smooth mortality rates across the classifications. The average of the mortality rates over the last two years of a classification is required to coincide with the average of mortality rates over the first two years of the next classification. A comparability ratio is defined as the sum of the probabilities of dying in the first two years of a new classification divided by the sum of the probabilities of dying in the last two years of the previous classification. The dates at which the countries adopted a new classification are presented in Appendix B, Table 14. In order to obtain data comparable over the complete period under observation, the number of deaths in a new classification is divided by the comparability ratio linking this classification with the previous one and previous comparability ratios where appropriate, that is

$$CompDeath_{x,c,t,d,s} =$$

$$\begin{cases} \frac{d_{x,c,t,d,s}}{CR_{x,c,d,s,j_{c,7},i_{c,8}}} & \text{for } i_{c,8} \leq t < j_{c,8} \\ \frac{d_{x,c,t,d,s}}{CR_{x,c,d,s,j_{c,7},i_{c,8}} \cdot CR_{x,c,d,s,j_{c,8},i_{c,9}}} & \text{for } i_{c,9} \leq t < j_{c,9} \\ \frac{d_{x,c,t,d,s}}{CR_{x,c,d,s,j_{c,7},i_{c,8}} \cdot CR_{x,c,d,s,j_{c,8},i_{c,9}} \cdot CR_{x,c,d,s,j_{c,9},i_{c,10}}} & \text{for } i_{c,10} \leq t < j_{c,10}, \end{cases} \qquad (9)$$

where

$CompDeath_{x,c,t,d,s}$ = number of persons of sex s, who die at age x, in country c, and year t, because of cause of death d, adjusted by the comparability ratios.

Most of these ratios take a value between 0.7 and 1.3. They are extremely close to one for cancer and the external causes of death. The higher and smaller values are usually at young and older ages.

Jumps in the mortality rates at the junction points between two classifications have been removed by these comparability ratios. The following analysis is applied to such mortality rates.

4 Long-Run Equilibrium among Mortality Rates

The procedure described in Section 2 is applied to our data set, two main objectives being sought. In Section 4.1, cointegrated relations between age-based risk factors in the United States are described and used in a new cause-specific mortality forecasting approach (Section 4.2). In Section 4.3, we look at the cointegrated relations between the five main causes of death in order to get a better understanding of the dependence existing between these causes.

4.1 Cointegrated Relations among Age-Based Risk Factors

In a paper of 1989, McNown and Rogers apply the Heligman-Pollard model on mortality rates in the United States over the period of 1900–1985. A few years later, in 1992, they apply the multi-exponential model on cause-specific mortality rates in the United States over the period 1960–1985. In both papers, they model the resulting time series of the parameters with ARIMA processes and use these models to forecast the value of the parameters and so, the complete age profile of mortality. However, both papers mention that *the univariate models yield independent projections of each parameter, ignoring potential gains in accuracy arising from taking into account the covariation among the (...) parameters.*

We provide an alternative approach, using multivariate time series, and thus, we include in the model time dependency and long-run trends that exist between the parameters. We choose to apply the Heligman-Pollard function on cause-specific mortality rates and to use VECM to model the parameters of the function as stochastic factors. The Heligman-Pollard function is a concise representation of mortality by age, each parameter having convenient demographic meaning (see Appendix C for a detailed description), and is defined as

$$\begin{aligned} q_{x,t} &= A_t^{(x+B_t)^{C_t}} + D_t e^{-E_t(\ln(x)-\ln(F_t))^2} \\ &\quad + \frac{G_t H_t^x}{1+K_t G_t H_t^x}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} q_{x,t} &= \text{probability of dying at age } x\text{, in year } t; \\ p_{x,t} &= 1 - q_{x,t}; \\ Z_t &= \text{value of the parameter } Z \text{ at time } t, \\ &\quad \text{for } Z \in (A,B,C,D,E,F,G,H,K). \end{aligned}$$

The function is a sum of three terms (equation 10): the first one represents mortality rates during childhood; the second one, mortality at middle ages (the accident hump); and the last one, mortality rates at older ages. Thus, the parameters have interpretation as factors impacting specific age ranges. Even if the Heligman-Pollard model is *complex and involves a large number of parameters,* (its) *flexibility is necessary to capture the full curvature of the mortality profile and the changes in this profile over time. Although other model schedules may nominally seem to employ fewer parameters, they in fact do so only by ignoring some features of the age curve of mortality (...).*[2] Furthermore, as mentioned by McNown and Rogers [1992], the *use of parameterized model schedules (...) offers a powerful instrument for representing mortality by cause in a way that facilitates comparisons across time or population subgroups.*

We illustrate the methodology described in Section 2 with mortality rates due to the diseases of the circulatory system for females in the United States. As the shape of the age pattern over time remains unchanged, it allows some parameters of this function to be fixed. Indeed, initial fit of the Heligman-Pollard function over the period 1950–2005 indicates that parameters B and F stay relatively constant. As fixing these parameters does not substantially reduce the accuracy of fit of the model, parameter B is fixed at the value of one, while parameter F is fixed at its median value, that is 33.85. Hence, only seven parameters need to be included in a VECM analysis and thus to be forecasted in order to get the complete age profile of mortality rates.

Lag order selection AIC, HQ, SC and FPE tests are performed on the data. The four criteria have the smallest value at a lag order of one, even with a constant or a trend included in the VAR.

Unit root tests KPSS, ADF and PP tests reveal similar results. The only parameter clearly stationary according to the three tests is parameter C, even at a one and ten percent significance levels (KPSS test accepts the null hypothesis of stationarity at a ten percent significance level – and so at lower significance levels as well; ADF and PP tests reject the null hypothesis of non-stationarity at a one percent significance level – and so at higher significance levels as well).

Results are contradictory for parameter D. ADF and PP tests reject the null hypothesis of non-stationarity at a five percent significance level, but accept it at a one percent significance level, while KPSS test accepts the null hypothesis of stationarity at a one percent significance level but rejects it at a five percent. The five remaining parameters are non-stationary according to the three tests.

As it is not possible to clearly determine if D is stationary, two models are fitted, one considering only parameter C as stationary and one considering parameters C and D as stationary. Normality tests on the residuals of both models as well as autocorrelations among the residuals reveal that the best

[2]McNown and Rogers [1989]

results are obtained with parameter D regarded as non-stationary. We then only present the results related to this model. Thus, if some cointegrated relations exist, they will only link the six non-stationary parameters: A, D, E, G, H and K.

Table 2: Tests on the number of cointegrated relations from the Johansen's procedure, 1950–2005, Circulatory system, USA, females

(a) Trace test

r	Trace stat	Critical values			
		10%	5%	2.50%	1%
5	0.66	2.69	3.76	4.95	6.65
4	7.80	13.33	15.41	17.52	20.04
3	25.14	26.79	29.68	32.56	35.65
2	48.59	43.95	47.21	50.35	54.46
1	88.91	64.84	68.52	71.8	76.07
0	142.96	89.48	94.15	98.33	103.18

(b) Maximum-eigenvalue test

r	Eigen stat	Critical values			
		10%	5%	2.50%	1%
5	0.66	2.69	3.76	4.95	6.65
4	7.14	12.07	14.07	16.05	18.63
3	17.34	18.6	20.97	23.09	25.52
2	23.45	24.73	27.07	28.98	32.24
1	40.32	30.9	33.46	35.71	38.77
0	54.06	36.76	39.37	41.86	45.1

The trace statistic tests the null hypothesis of r cointegrated relations against the alternative of n cointegrated relations, where n corresponds to the number of variables under observation and $r < n$.
The maximum-eigenvalue statistic tests the null hypothesis of r cointegrated relations against the hypothesis of $r + 1$ cointegrated relations.
A null hypothesis is accepted at a $\alpha\%$ significance level when the statistic is lower than the corresponding critical value. Thus, these tables indicate that two cointegrated relations are accepted at a 2.5% significance level.
These two tests assess the number of long-run equilibrium relationships among the parameters of the Heligman-Pollard function (A, D, E, G, H and K), that is among age-based risk factors.

Cointegrated relations We applied the trace test and the maximum-eigenvalue test of the Johansen's procedure. Results are presented in Tables 2. The trace test compares the null hypothesis that there are r cointegrated relations against the alternative of n cointegrated relations, where n corresponds to the number of variables under observation and $r < n$. This test indicates that we accept the null hypothesis of two, three, four or five cointegrated relations against six cointegrated relations at a 2.5% significance level.

The maximum-eigenvalue statistic tests the null hypothesis of r cointegrated relations against the hypothesis of $r+1$ cointegrated relations. Table 2(b) indicates that the null hypothesis of two cointegrated relations is accepted, while the null hypothesis of one cointegrated relation is rejected. Hence, there are two long-run equilibrium relationships among the parameters of the Heligman-Pollard function, and thus, age-based risk factors are linked to each other through these two relations.

Fitted VECM Knowing the number of cointegrated relations, the Johansen's procedure can be applied in order to quantify these relations. Equation 8 is then fitted and the resulting model is

$$\begin{bmatrix} \nabla A_t \\ \nabla C_t \\ \nabla D_t \\ \nabla E_t \\ \nabla G_t \\ \nabla H_t \\ \nabla K_t \end{bmatrix} = \begin{bmatrix} 4.34 \cdot 10^{-4} \\ 0.12 \\ -4.03 \cdot 10^{-3} \\ 5.27 \cdot 10^{2} \\ 4.57 \cdot 10^{-5} \\ -0.91 \\ -6.62 \cdot 10^{2} \end{bmatrix} + \begin{bmatrix} -1.97 \cdot 10^{-6} & 8.99 \cdot 10^{-8} \\ -2.98 \cdot 10^{-3} & -3.03 \cdot 10^{-4} \\ 3.02 \cdot 10^{-6} & -2.89 \cdot 10^{-6} \\ -0.43 & 0.37 \\ -2.61 \cdot 10^{-7} & 2.24 \cdot 10^{-9} \\ 1.61 \cdot 10^{-3} & -5.26 \cdot 10^{-4} \\ 0.69 & -0.45 \end{bmatrix}$$

$$\times \begin{bmatrix} -1.02 \cdot 10^{4} & 8.50 \cdot 10^{4} & -0.39 & -1.73 \cdot 10^{6} & -1.41 \cdot 10^{2} & 0.15 \\ -6.52 \cdot 10^{3} & -1.52 \cdot 10^{5} & -0.36 & 2.79 \cdot 10^{6} & 1.08 \cdot 10^{3} & -1.31 \end{bmatrix}$$

$$\times \begin{bmatrix} A_{t-1} \\ D_{t-1} \\ E_{t-1} \\ G_{t-1} \\ H_{t-1} \\ K_{t-1} \end{bmatrix}. \quad (11)$$

The multiplicative part of the second element of the right hand side of equation 11 represents the two cointegrations between the parameters. We may write them as

$$\begin{aligned} z_{1t} = & - 1.02 \cdot 10^4 A_t + 8.50 \cdot 10^4 D_t - 0.39 E_t - 1.73 \cdot 10^6 G_t - 1.41 \cdot 10^2 H_t \\ & + 0.15 K_t, \\ z_{2t} = & - 6.52 \cdot 10^3 A_t - 1.52 \cdot 10^5 D_t - 0.36 E_t + 2.79 \cdot 10^6 G_t + 1.08 \cdot 10^3 H_t \\ & - 1.31 K_t, \end{aligned} \quad (12)$$

where z_{1t} and z_{2t} are two stochastic variables representing the deviation from the equilibrium. These two variables should be stationary. The sign associated with each parameter is of real interest in these relations.[3] According to the first cointegrated relation, if, for example, mortality around age zero decreases (that is parameter A decreases), then, in order to counter this decline

[3] The size of the coefficients is influenced by the value taken by the parameters. For example, as parameters A, D and G are small, their associated coefficients are high.

Table 3: Tests on residuals of the fitted VECM, 1950–2005, Circulatory system, USA, females

Type of test	**Name of the test**	**Statistic value**	**p-value**
Autocorrelation	Portmanteau (15 lags)	776.68	0.07
	Portmanteau (25 lags)	1224.33	0.39
Normality	Skewness	8.39	0.30
	Kurtosis	8.05	0.33
	Both	16.44	0.29

The Portmanteau statistic tests the residual autocorrelations up to 15 and 25 lags, that is the null hypothesis of no-autocorrelation up to 15 or 25 lags.

Three normality tests are performed on the residuals. The first one is based on the skewness statistic, the second one on the kurtosis statistic and the last one, called *both*, is a combination of the first two.

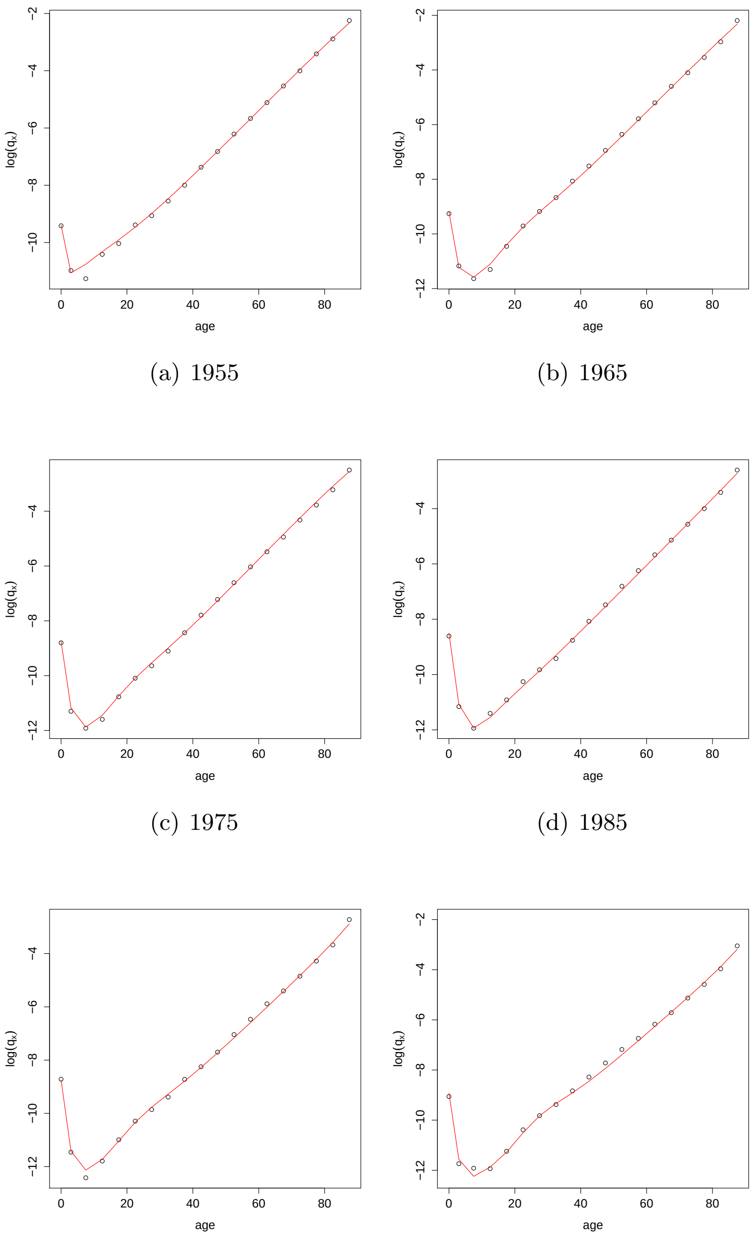
for the relation to keep stationary, either the accident hump will decrease or have an impact on a smaller age range (that is parameter D will decline or E will increase respectively) or mortality for the elderly will growth (parameter K will decline; parameter G or parameter H will increase), or a combination of these impacts.

Besides, an increase in mortality at older ages is reflected on the Heligman-Pollard function with an increase in parameter G or in parameter H or a decrease in parameter K, depending on the type of mortality increase (details in Appendix C). This relation is included in both cointegrations as parameters G and H have an associated coefficient with similar sign, while the coefficient of K is of opposite sign. The mortality rates induced from this model are presented in Figure 2, along with the data set.

Model validation Finally, the residuals of the model need to be checked for normality as well as no-autocorrelation as theories on VAR and VECM are based on such assumptions. The computed statistics are summarized in Table 3. The Portmanteau test is a popular test for the overall significance of the residual autocorrelations up to lag l. The Portmanteau statistic has an approximate asymptotic Chi-square distribution for large value of l. Thus, we test the null hypothesis of no-autocorrelation among the residuals up to $l = 15$ and $l = 25$ lags. The statistic used in this paper is the Portmanteau statistic adjusted for small sample.[4] As indicated by the p-values, the null hypothesis is twice accepted at a five percent significance level.

Three tests for normality are based on the third and fourth central mo-

[4]As mentioned by Lütkepohl [2005], some researchers found that in small samples the test has a low power against many alternatives. Hence, a modified test statistic has been suggested.



(a) 1955 (b) 1965

(c) 1975 (d) 1985

(e) 1995 (f) 2005

Figure 2: Sample years of the log-mortality rates and the fitted VECM, Circulatory system, USA, females

The dots represent the observed mortality rates, while the fitted model is depicted by the curve.

ments (skewness and kurtosis) of a normal distribution.[5] The test statistic called *both* in Table 3 is a joint test of the former two. The three tests clearly state that the null hypothesis of normality is accepted. Thus, the residuals of the fitted model on the parameters of the Heligman-Pollard function are normally distributed and no-autocorrelation remains. The model catches then accurately the age dependance of mortality rates due to the circulatory system in the United States and is a good representation of the data generating process.

4.2 Projections

As the fitted model in the previous section manages to catch the features of the data with few parameters (only 33 parameters for 19 age-groups over 56 years, that is 1'064 observations), we may wonder if using such a model gives accurate forecasts. For that purpose, the model is fitted on a shorter period and projections are performed for the remaining years such that we may compare the forecasts with the actual data. Many experts agree that at least 50 years of observation are needed in order to conduct a VECM analysis. Therefore, the model is fitted over the period 1950–2000, following the same procedure as the one described in previous sections. Parameter B is still fixed at a value of one and the median value of parameter F over the period 1950–2000 is 25.42. Forecasts are done for 20 years. As data are only available until 2005, the first five years of forecasted mortality rates are compared with the actual data.

To assess the accuracy of the projections, two additional models are fitted over the period 1950–2000 and used to forecast mortality rates until 2005. The first one is the well-known Lee-Carter model, which has been successfully applied at the population level for the US data in the past. The second one fits the Heligman-Pollard function on mortality rates and uses univariate time series, the traditional ARIMA processes, in order to model the parameters. This method has been already applied by McNown and Rogers (McNown and Rogers [1989] and McNown and Rogers [1992]).

4.2.1 VECM

AIC, HQ, SC and FPE tests are performed on the data over the period 1950–2000. As previously, the four criteria have the smallest value at a lag order of one.

Tests on the stationarity of the parameters are also carried out. ADF and PP tests reject the null hypothesis of non-stationarity at a one percent significance level for parameters C, D and E, while KPSS test accepts the null hypothesis of stationarity at a ten percent significance level only for parameter C and at a one percent significance level for parameter E. In the

[5]For a detailed description of these tests, see Lütkepohl [2005].

following analysis, we assume that parameters C, D and E are stationary as at least two out of the three tests suggest it.

Table 4: Tests on the number of cointegrated relations from the Johansen's procedure, 1950–2000, Circulatory system, USA, females

(a) Trace test

r	Trace stat	Critical values			
		10%	5%	2.50%	1%
3	4.27	2.69	3.76	4.95	6.65
2	12.71	13.33	15.41	17.52	20.04
1	40.04	26.79	29.68	32.56	35.65
0	88.70	43.95	47.21	50.35	54.46

(b) Maximum-eigenvalue test

r	Eigen stat	Critical values			
		10%	5%	2.50%	1%
3	4.27	2.69	3.76	4.95	6.65
2	8.44	12.07	14.07	16.05	18.63
1	27.34	18.6	20.97	23.09	25.52
0	48.66	24.73	27.07	28.98	32.24

The trace statistic tests the null hypothesis of r cointegrated relations against the alternative of n cointegrated relations, where n corresponds to the number of variables under observation and $r < n$.
The maximum-eigenvalue statistic tests the null hypothesis of r cointegrated relations against the hypothesis of $r + 1$ cointegrated relations.
A null hypothesis is accepted at a $\alpha\%$ significance level when the statistic is lower than the corresponding critical value. Thus, these tables indicate that two cointegrated relations are accepted at a 2.5% significance level.
These two tests assess the number of long-run equilibrium relationships among the parameters of the Heligman-Pollard function (A, G, H and K), that is among age-based risk factors.

Two cointegrated relations are found among the remaining four parameters, as indicated in Table 4. The resulting VECM is

$$\begin{bmatrix} \nabla A_t \\ \nabla C_t \\ \nabla D_t \\ \nabla E_t \\ \nabla G_t \\ \nabla H_t \\ \nabla K_t \end{bmatrix} = \begin{bmatrix} 2.13 \cdot 10^{-3} \\ -4.25 \\ 4.79 \cdot 10^{-4} \\ -6.60 \cdot 10^{2} \\ -1.59 \cdot 10^{-4} \\ 0.79 \\ -3.65 \cdot 10^{2} \end{bmatrix} + \begin{bmatrix} -2.07 \cdot 10^{-6} & 1.61 \cdot 10^{-6} \\ -2.72 \cdot 10^{-3} & -1.22 \cdot 10^{-3} \\ 2.74 \cdot 10^{-7} & 1.46 \cdot 10^{-7} \\ -5.88 \cdot 10^{-2} & -0.30 \\ -2.07 \cdot 10^{-7} & -1.44 \cdot 10^{-8} \\ 9.11 \cdot 10^{-4} & 1.05 \cdot 10^{-4} \\ 0.24 & -0.24 \end{bmatrix}$$

$$\times \begin{bmatrix} -8.12 \cdot 10^{3} & 3.61 \cdot 10^{5} & 5.50 \cdot 10^{2} & -0.65 \\ 3.44 \cdot 10^{4} & 5.44 \cdot 10^{6} & 1.86 \cdot 10^{3} & -2.27 \end{bmatrix} \times \begin{bmatrix} A_{t-1} \\ G_{t-1} \\ H_{t-1} \\ K_{t-1} \end{bmatrix}, \qquad (13)$$

with a detailed description of the cointegrations in equations 14. Appendix

D presents graphs of the parameter history as well as the fitted model and the projected values. As for the cointegrated relations of the previous model fitted over the period 1950–2005, parameters G and H have an associated coefficient with similar sign, while the coefficient of K is of opposite sign, as shown in

$$\begin{aligned} -8.12 \cdot 10^3 A_t + 3.61 \cdot 10^5 G_t + 5.50 \cdot 10^2 H_t - 0.65 K_t &= z_{1t}, \\ 3.44 \cdot 10^4 A_t + 5.44 \cdot 10^6 G_t + 1.86 \cdot 10^3 H_t - 2.27 K_t &= z_{2t}. \end{aligned} \quad (14)$$

It is, once again, in line with their demographic interpretation.

Normality tests as well as tests on the autocorrelations among the residuals are performed (Table 5). The normality of the residuals is accepted at a five percent significance level. However, according to the Portmanteau test, some autocorrelations among the residuals remain. A VECM with higher order, such as two or three lags, may be better. As the main purpose in this section is to build a forecasting model, we prefer to have a model with as few parameters as possible as slight under-fitting may even improve the forecasting performance.

Table 5: Tests on residuals of the fitted VECM, 1950–2000, Circulatory system, USA, females

Type of test	**Name of the test**	**Statistic value**	**p-value**
Autocorrelation	Portmanteau (15 lags)	872.36	$8.70 \cdot 10^{-5}$
	Portmanteau (25 lags)	1451.54	$2.01 \cdot 10^{-6}$
Normality	Skewness	12.19	0.09
	Kurtosis	11.02	0.14
	Both	23.21	0.06

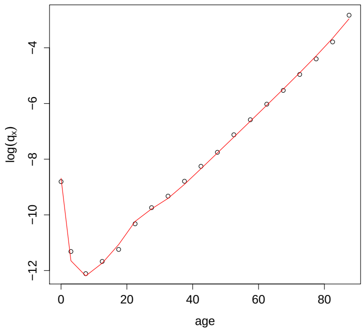
The Portmanteau statistic tests the residual autocorrelations up to 15 and 25 lags, that is the null hypothesis of no autocorrelation up to 15 or 25 lags.

Three normality tests are performed on the residuals. The first one is based on the skewness statistic, the second one on the kurtosis statistic and the last one, called *both*, is a combination of the first two.

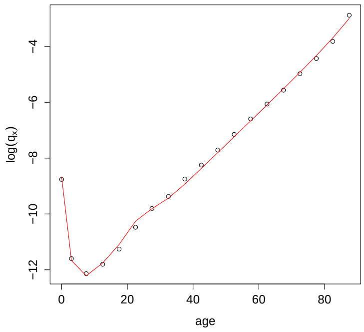
In Figure 3, the forecasted mortality rates (curve) are compared with the actual data (dots), and Figure 4 describes the observed mortality rates in 2000 (dots) along with a few forecasts. The model follows accurately the data. A comparison with other well-known models will reveal if a model using a VECM analysis not only provides an accurate description of the data, but also offers better performance.

4.2.2 Lee-Carter Model / ARIMA Processes

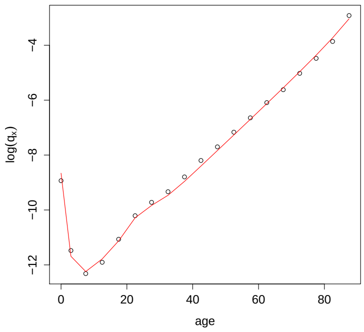
The forecasted pattern of mortality generated by the VECM applied to the parameters of the Heligman-Pollard function is compared with two alter-



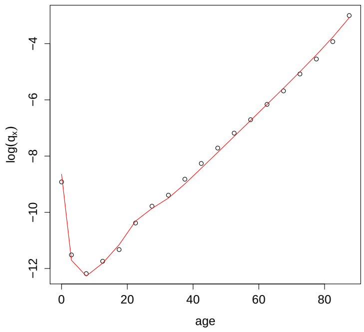
(a) 2001



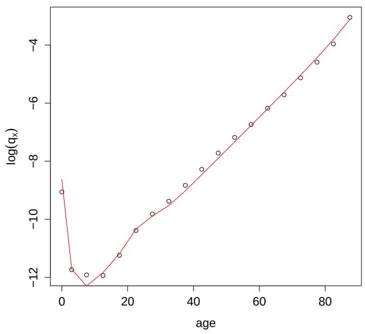
(b) 2002



(c) 2003



(d) 2004



(e) 2005

Figure 3: Sample years of the log-mortality rates and the forecasted values, Circulatory system, USA, females

The dots represent the observed mortality rates, while the forecasted values are depicted by the curve.

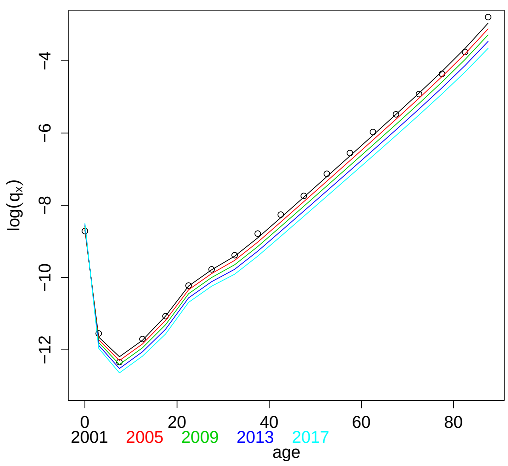


Figure 4: Forecasted mortality rates over the period 2001–2020, Circulatory system, USA, females

The actual mortality rates of 2000 are represented by the dots and the forecasts by the curves.

natives. The first one refers to the Lee-Carter model, which has become a standard in mortality modeling (Lee and Carter [1992]). The main idea in this model is to decompose the logarithm of the force of mortality in two components: one describing the age pattern of average mortality rates; the other for a common time trend with differential impacts by age. The model is defined as

$$\ln \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{15}$$

with

$\mu_{x,t}$	=	force of mortality at age x, and year t;
α_x	=	mean value over time, at age x, of the logarithm of the force of mortality;
β_x	=	deviation from the mean value α_x at age x. It reflects the impact of the time trend represented by κ_t on age, that is the higher its absolute value, the more important are the changes of mortality between two years;
κ_t	=	mortality rates trend over time;
$\epsilon_{x,t}$	=	historical influences not captured by the model;
	=	errors with mean zero and variance σ^2 (homoscedasticity).

To allow this interpretation of the parameter α_x, the sum over the estimator $\hat{\kappa}_t$ is set equal to zero,

$$\sum_{t=t_{min}}^{t_{max}} \hat{\kappa}_t = 0. \tag{16}$$

Indeed, if N is the number of years under observation,

$$\begin{aligned}\prod_{t=t_{min}}^{t_{max}} \mu_{x,t} &= \exp\left(\hat{\alpha}_x N + \hat{\beta}_x \sum_{t=t_{min}}^{t_{max}} \hat{\kappa}_t\right) \\ &= \exp\left(\hat{\alpha}_x N\right),\end{aligned}$$

which leads to

$$\hat{\alpha}_x = \frac{1}{N}\sum_{t=t_{min}}^{t_{max}} \ln \mu_{x,t}, \qquad \forall \quad x. \tag{17}$$

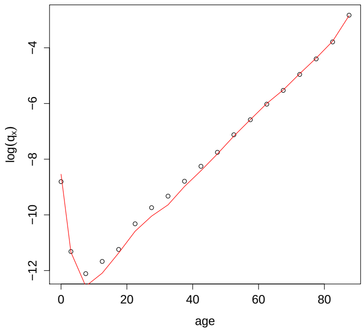
In order to have an identifiable model, another constraint on the parameters is specified, which usually is

$$\sum_{x=x_{min}}^{x_{max}} \hat{\beta}_x = 1. \tag{18}$$

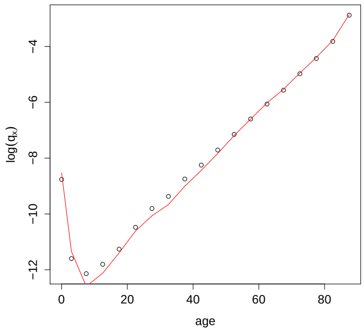
As suggested by Delwarde and Denuit [2006], equation 15 is fitted by maximum likelihood estimation, assuming that the number of deaths at age x follows a Poisson distribution with mean $l_{x,t} \cdot \mu_{x,t}$, $l_{x,t}$ being the population of age x at the beginning of year t. The Newton-Raphson iterations are used for the estimation of the parameters, as described in Delwarde and Denuit [2006]. The resulting parameters are introduced in Appendix E.

In the Lee-Carter model, a single common factor across ages is used for determining the general level of mortality improvement over time, and thus, a single time series (κ_t) needs to be projected in order to forecast the complete age profile of mortality. Lee and Carter [1992] suggest the use of a simple random walk with a drift, that is an ARIMA(0,1,0) model. Following the method proposed by Pandit and Wu [2001], we also conclude that an ARIMA(0,1,0) process with a drift of -0.36 is the best model. The resulting forecasts associated with the actual mortality rates are introduced in Figure 5.

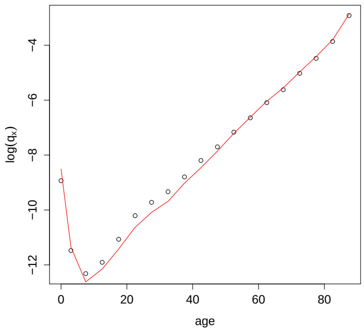
In the second alternative, the parameters of the Heligman-Pollard function fitted over the period 1950–2000 are modeled with the traditional ARIMA processes, instead of a VECM. First differencing transformation is performed on every parameter to assure stationarity. Appendix F reports the final ARIMA models for the seven non-fixed parameters. As for parameter κ_t of the Lee-Carter model, the procedure described in Pandit and Wu [2001] is



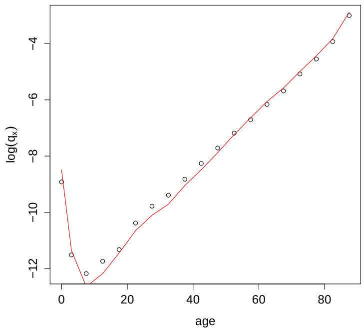
(a) 2001



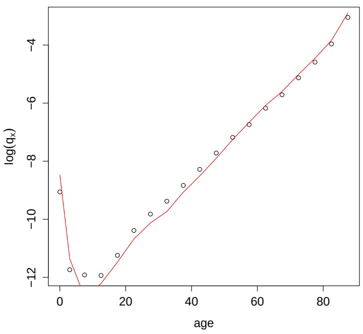
(b) 2002



(c) 2003



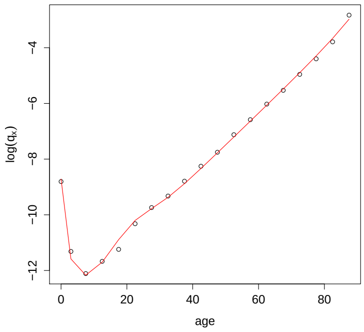
(d) 2004



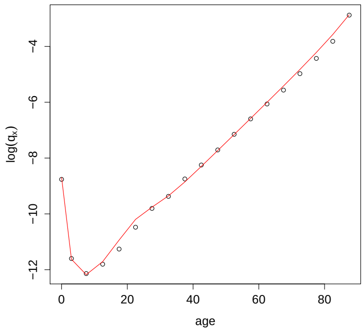
(e) 2005

Figure 5: Sample years of the log-mortality rates and the forecasted values according to the Lee-Carter model, Circulatory system, USA, females

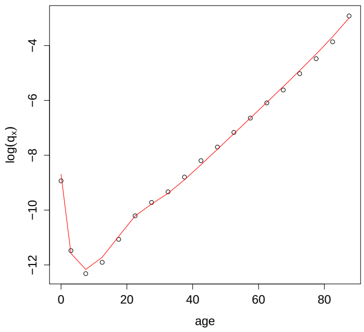
The dots represent the observed mortality rates, while the forecasted values from the Lee-Carter model are depicted by the curve.



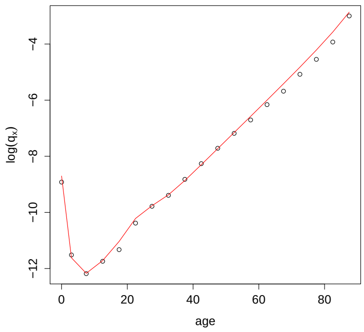
(a) 2001



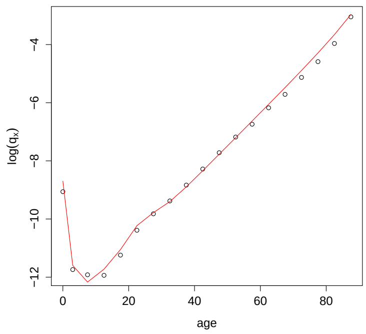
(b) 2002



(c) 2003



(d) 2004



(e) 2005

Figure 6: Sample years of the log-mortality rates and the forecasted values according to ARIMA models, Circulatory system, USA, females

The dots represent the observed mortality rates, while the forecasted values – using the Heligman-Pollard function and forecasting the parameters with ARIMA processes – are depicted by the curve.

followed. Each model shows no significant residual autocorrelation according to the Portmanteau test. Figure 6 contains the resulting forecasted age profiles of mortality.

The plots of mortality rates according to the three different approaches of projection (Figures 3, 5 and 6) show that the Lee-Carter model tends to over-estimate the decline of mortality at young ages, while the Heligman-Pollard function associated with ARIMA processes tends to underestimate mortality improvement, particularly at older ages.

The forecasting performance of the three methods is further evaluated with two summary statistics, assessing the match between the actual and the forecasted mortality rates from 2001 to 2005. The mean absolute percentage error statistic (MAPE) is the average of the absolute percentage error between the forecasted and the observed mortality rates, computed over the nineteen age-groups. The second statistic compares the forecasted mortality with a standard: the no-change forecast. The no-change forecast takes the mortality of 2000 as fixed and as being the forecasted mortality rates of the following five years. Our second statistic is the ratio of the square root of two mean square errors (MSE) computed over the nineteen age-groups. The first MSE is computed between the forecasted and the observed mortality rates. The second MSE is between the no-change forecast and the observed mortality rates. Thus, a statistic with a value smaller than unity means that the model forecasts more accurately than the base assumption of no change in mortality. Table 6 summarizes the results.

Table 6: Mean absolute percentage error and ratio of the mean square error between one of the three forecasting models and the no-change forecast, Circulatory system, USA, females

Year	VECM		Lee-Carter		ARIMA	
	MAPE	vs no-change forecast	MAPE	vs no-change forecast	MAPE	vs no-change forecast
2001	9.56%	3.21	13.04%	0.44	9.90%	3.51
2002	9.33%	1.27	14.15%	0.30	11.08%	1.25
2003	11.71%	0.87	17.85%	0.40	10.71%	0.78
2004	11.84%	0.43	19.33%	0.55	15.37%	0.97
2005	13.46%	0.33	24.03%	0.57	17.06%	0.59

MAPE is the mean absolute percentage error. The *vs no-change forecast* is the ratio of the square root of the mean square error of one of the three considered models with the square root of the mean square error of the no-change forecast. The no-change forecast takes the mortality of 2000 as fixed and as being the forecasted mortality rates of the following five years.

All summary statistics are averages over the nineteen age-groups and describe the situation at a single-point in time.

The forecasts with the VECM are closer to the actual mortality rates than the forecasts of the Lee-Carter model or the forecasts performed with the Heligman-Pollard function and ARIMA processes. The forecast error (MAPE) with the VECM is lower and increases more slowly with time than the other two models, and thus, we may expect that for projections over longer time horizon, VECM will perform better.

The principal benefit of the Heligman-Pollard function over the Lee-Carter model is that it does take advantage of the strong constancy observed in the age profile of mortality rates. Thus, as the Heligman-Pollard function enforces a particular age pattern on its forecasts, results can less easily be unrealistic. However, the Lee-Carter model is more accurate at older ages during the first forecasted years. As mortality rates at older ages are higher than rates at young and middle ages, the absolute error is higher at older ages as well. As the statistic that compares one model with the no-change forecast uses the square error, the statistic for the Lee-Carter model is lower for the first three years than the statistic of the other two models. However, it does not last and the Heligman-Pollard (HP) model associated with the VECM is quickly superior. As the forecast interval is extended, the age pattern imposed by the structure of the Heligman-Pollard function produces an improvement in accuracy, which is enforced by the VECM taking into account the dependencies between the parameters of the function. We even expect the association HP - VECM to be more precise for longer forecast periods. However, data over longer time horizon are needed to confirm it.

4.3 Cointegrated Relations among Causes of Death

Correlations between competing risks are important matters, for which there is no current solution. The usual assumption is that causes of death are independent when there are considered. However, we know that such an assumption does not correspond to reality and may spoil the results of a mortality risk model. This section reveals important features about the relations binding the five main causes of death, improving our understanding of the dependance between these competing risks.

For that purpose, a *global* mortality rate is used, which is a mortality rate not disaggregated by age, that is

$$q_{c,t,d,s} = d_{c,t,d,s}/l_{c,t,s},$$

where

$$\begin{aligned} q_{c,t,d,s} &= \text{probability of dying in country } c\text{, at time } t\text{,} \\ &\quad\ \text{from cause of death } d\text{, and for a person of sex } s\text{;} \\ d_{c,t,d,s} &= \text{number of persons of sex } s\text{, dying in country } c\text{, at time } t\text{,} \\ &\quad\ \text{from cause of death } d\text{;} \\ l_{c,t,s} &= \text{number of persons of sex } s\text{, in country } c\text{,} \\ &\quad\ \text{alive at the beginning of year } t\text{.} \end{aligned}$$

As the age structure of the population changes over time, it affects such a *global* mortality rate. Thus, we define $q^*_{c,t,d,s}$, $d^*_{c,t,d,s}$ and $l_{c,LY_c,s}$ as

$$\begin{aligned} q^*_{c,t,d,s} &= d^*_{c,t,d,s} / l_{c,LY_c,s}, \\ d^*_{c,t,d,s} &= \sum_x q_{x,c,t,d,s} \times l_{x,c,LY_c,s}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} q_{x,c,t,d,s} &= \text{probability of dying in country } c\text{, at time } t\text{,} \\ & \text{from cause of death } d\text{, for a person of sex } s\text{, and age } x\text{;} \\ l_{x,c,LY_c,s} &= \text{number of persons of sex } s\text{, and age } x\text{, alive in country } c\text{,} \\ & \text{at the beginning of year } LY_c\text{;} \\ l_{c,LY_c,s} &= \sum_x l_{x,c,LY_c,s}; \\ &= \text{number of persons of sex } s\text{, in country } c\text{,} \\ & \text{alive at the beginning of year } LY_c\text{;} \\ LY_c &= \text{last year under observation for country } c. \end{aligned}$$

We use the population of the last year under observation as a base. We compute the total number of deaths in a particular year t as if the population alive at the beginning of that year was the same as the population of the last year under observation. Thus, the $q^*_{c,t,d,s}$ refers to the *global* mortality rate in year t, assuming that the population is constant during the complete period under observation and fixed at the level of the last observed year.

The VECM analysis is applied across nine major countries for males and females. Long-run equilibrium relationships are found between the five main causes of death and similarities between countries are described below. The following analysis is applied on the logarithm of $q^*_{c,t,d,s}$.

Lag order selection Out of the four tests performed, at least two of them, if not all of them, indicate a lag order of one as optimal. Thus, a VAR(1) is the most suitable model in every of the nine analyzed countries.

Unit root tests KPSS, ADF and PP tests are performed on the data. A cause of death is said stationary when at least two out of the three tests accept it at a five percent significance level. When some doubts still remain, several models are tested and the one with non-autocorrelated and normally distributed residuals is preferred. Table 7 presents the causes of death retained as stationary according to this procedure. In most of the countries, either all causes of death are non-stationary or only the diseases of the respiratory system are stationary. Indeed, in the United States, Australia, Italy (females only), Sweden (females) and United Kingdom (males), the five main

Table 7: Stationarity of the five main causes of death in nine countries

	Males	Females
USA	All causes: UR	All causes: UR
Australia	All causes: UR	All causes: UR
Switzerland	Respiratory: S Other causes: UR	Cancer, Respiratory: S Other causes: UR
Japan	Respiratory: S Other causes: UR	Respiratory: S Other causes: UR
Singapore	I&P: S Other causes: UR	I&P: S Other causes: UR
Italy	Respiratory: S Other causes: UR	All causes: UR
Norway	Respiratory: S Other causes: UR	Respiratory: S Other causes: UR
Sweden	Respiratory: S Other causes: UR	All causes: UR
United Kingdom	All causes: UR	Respiratory: S Other causes: UR

UR = Unit root, that is a non-stationary variable; S = Stationary variable; I&P = Infectious and parasitic diseases.

This table describes the stationarity of the log-mortality rate $\ln q^*_{c,t,d,s}$. A variable is said to be stationary when at least two out of the three tests (that is KPSS, ADF and PP tests) accept it at a five percent significance level or when it provides the best model according to the model validation criteria.

causes of death are non-stationary. On the other hand, in Switzerland (males only), Japan, Italy (males), Norway, Sweden (males) and United Kingdom (females), the diseases of the respiratory system are the only stationary variable. Singapore reveals completely different results: Infectious and parasitic diseases are the only stationary cause of death. The shorter period under observation as well as the climate of this country may explain this. Indeed, Singapore is the only country for which less than 50 years are observed and is also the only country with a tropical weather.

Cointegrated relations The number of cointegrated relations is summarized in Table 8. These results follow from the trace and maximum-eigenvalue tests of the Johansen's procedure at a five percent significance level. However, for females in Australia and United Kingdom, the cointegration is accepted at a 2.5% significance level. Furthermore, results of the two tests are difficult to interpret for females in Singapore, Norway and Sweden. Several models are then tested and the most efficient one according to the model validation criteria (non-autocorrelated and normally distributed residuals) is presented in Table 8. An interesting comment should be raised: In most countries, particularly in Europe, only one long-run equilibrium relationship exists.

Fitted VECM Appendices G and H contain the fitted VECM for every country and both sexes, according to the stationarity described in Table 7 and the number of cointegrated relations presented in Table 8. Some common features between countries, worth to be raised, are described in Tables 9 and 10.

Countries with the diseases of the respiratory system stationary are classified in two categories (Table 9). Females in Japan as well as males in Italy, Norway and Sweden show similar experience. Indeed, the diseases of the circulatory system, cancer and the infectious and parasitic diseases have a coefficient with similar sign, while the coefficient of the external causes of death is of opposite sign. In these four cases, the long-run equilibrium relationship is similar and thus, these countries had similar behavior in the past. For example, if the diseases of the circulatory system increased, either cancer or the infectious and parasitic diseases decreased, or the external causes of death increased for the relation to keep stationary.

The second group of countries displaying similar behavior includes males in Switzerland and Japan as well as females in United Kingdom. Here, the diseases of the circulatory system and the infectious and parasitic diseases have a coefficient with similar sign, while the coefficients of cancer and of the external causes of death are of opposite sign. Thus, if the diseases of the circulatory system increased, either cancer or the external causes of death increased, or the infectious and parasitic diseases decreased for the relation

Table 8: Number of cointegrated relations among the five main causes of death in nine countries

	Males	Females
USA	2	1
Australia	3	1
Switzerland	1	0
Japan	1	3
Singapore	0	1
Italy	1	1
Norway	1	1
Sweden	1	1
United Kingdom	2	1

Number of cointegrated relations according to the trace and maximum-eigenvalue tests of the Johansen's procedure at a five percent significance level, except for females in Australia and in United Kingdom as one cointegrated relation is accepted at a 2.5% significance level. For females in Singapore, Norway and Sweden, several models are tested and the table reports the best model according to the model validation criteria.

Table 9: Long-run equilibrium relationships in countries with similar experience, the diseases of the respiratory system being stationary

		Circulatory system	Cancer	External causes	Infectious and parasitic diseases
Switzerland - Males		-14.90	17.52	17.78	-0.64
Japan	Males	0.99	-4.61	-3.33	1.20
	Females	-2.42	-27.85	21.31	-2.98
Italy - Males		8.42	15.32	-18.75	2.40
Norway - Males		-6.80	-8.81	4.31	-2.52
Sweden - Males		12.08	16.98	-13.21	3.88
UK - Females		-2.71	2.08	1.32	-1.53

One VECM is fitted by country following the Johansen's procedure. The variables used in a VECM are the five main causes of death of the country under study. Some of the cointegrated relations included in the model corresponding to a particular country are described in this table (all the cointegrated relations are presented in Appendix G, Table 18). For example, a VECM representing male mortality in Switzerland uses one long-run equilibrium relationship (cointegrated relation), that we may write as

$$-14.90 \times CircSyst_t + 17.52 \times Cancer_t + 17.78 \times ExtCauses_t - 0.64 \times I\&P_t = z_t,$$

where z_t is a stationary variable.

Table 10: Long-run equilibrium relationships in countries with similar experience, all causes of death being non-stationary, males

	Circulatory system	Cancer	Respiratory system	External causes	Infectious and parasitic diseases
USA	1.04	-2.35	-0.42	-6.96	-2.16
Australia	-1.91	11.45	1.62	-1.12	0.50
	18.50	-20.38	-0.69	-28.33	-1.37
UK	4.55	-11.08	-1.43	3.58	-0.75
	1.31	-25.36	-6.92	18.39	-4.22

One VECM is fitted by country following the Johansen's procedure. The variables used in a VECM are the five main causes of death of the country under study. Some of the cointegrated relations included in the model corresponding to a particular country are described in this table (all the cointegrated relations are presented in Appendix G, Table 18). For example, a VECM representing male mortality in the United States uses one long-run equilibrium relationship (cointegrated relation), that we may write as

$$1.04 \times CircSyst_t - 2.35 \times Cancer_t - 0.42 \times RespSyst_t - 6.96 \times ExtCauses_t - 2.16 \times I\&P_t = z_t,$$

where z_t is a stationary variable.

to still be at an equilibrium.

Countries with all causes of death non-stationary are as well classified in two categories, these two groups only summarizing male behaviors (no common experience is found for females, Table 10). The two long-run equilibriums for males in United Kingdom correspond to one of the two relations presented in Table 10 for Australia. An increase in the diseases of the circulatory system in these two countries implied an increase in cancer, in the diseases of the respiratory system or in the infectious and parasitic diseases, or a decrease in the external causes of death. Besides, the remaining cointegrated relation presented in Table 10 for Australia is similar to the relation introduced in Table 10 for the United States. An increase in the diseases of the circulatory system was followed by an increase in one or a combination of the four remaining causes.

In summary, it appears that groups of countries have similar behaviors, especially for males. This information is of primary importance and should be considered in a risk diversification approach.

Model validation Residuals of the fitted models need to be checked for normality and no-autocorrelation. Tables 11 summarize the significance of the tests for males and females. The null hypothesis of normality as well as the null hypothesis of no-autocorrelation up to 15 or 25 lags are, in most cases, accepted at a five percent significance level. Only for males in Italy as well as females in Singapore and United Kingdom, the kurtosis test and the joint test of the kurtosis and skewness ones (called *both* in Tables 11) reject the null hypothesis of normality.

Table 11: Tests on residuals of the fitted VECM on causes of death

(a) Males

	Portmanteau test		Normality tests		
	15 lags	25 lags	skewness	kurtosis	both
USA	***	***	***	***	***
Australia	***	**	***	***	***
Switzerland	***	***	***	***	***
Japan	***	***	***	***	**
Singapore	***	***	***	***	***
Italy	***	***	***	—	—
Norway	*	*	***	***	***
Sweden	***	***	***	***	***
UK	***	***	***	***	***

* The null hypothesis is accepted at a one percent significance level.
** The null hypothesis is accepted at a 2.5% significance level.
*** The null hypothesis is accepted at a five percent significance level.
– The null hypothesis is rejected.

Hence, the VECM described in this section manage to capture key features of the cause of death data and improve our understanding of the dependance between these competing risks. The long-run equilibrium relationships are reliable and should not be disregarded in any analyses considering the causes of death.

5 Conclusion

Our study demonstrates the use of multivariate dynamic systems to model mortality rates in two different settings. In the first place, a parametrized mortality model, the Heligman-Pollard function, is fitted to cause-specific mortality rates in the United States over the period 1950–2000. The resulting time series of the parameters is forecasted using VECM. Such models incorporate common stochastic trends that exist between the variables,

Table 11: Tests on residuals of the fitted VECM on causes of death - continue

(b) Females

	Portmanteau test		Normality tests		
	15 lags	25 lags	skewness	kurtosis	both
USA	***	**	***	***	***
Australia	***	***	***	**	***
Switzerland	***	***	***	***	***
Japan	***	***	***	***	***
Singapore	***	***	***	—	—
Italy	***	***	***	***	***
Norway	***	***	***	***	***
Sweden	***	***	***	***	***
UK	***	***	***	—	—

* The null hypothesis is accepted at a one percent significance level.
** The null hypothesis is accepted at a 2.5% significance level.
*** The null hypothesis is accepted at a five percent significance level.
– The null hypothesis is rejected.
The Portmanteau statistic tests the residual autocorrelations up to 15 and 25 lags, that is the null hypothesis of no-autocorrelation up to 15 or 25 lags.

Three normality tests are performed on the residuals. The first one is based on the skewness statistic, the second one on the kurtosis statistic and the last one, called *both*, is a combination of the first two.

along with long-run equilibrium relationships. Thus, they capture the age dependance of cause-specific mortality rates. Indeed, the parameters of the Heligman-Pollard function have interpretation as factors impacting specific age ranges.

Our analysis reveals that a forecasting model based on the Heligman-Pollard function associated with VECM is a significant improvement over the well-known Lee-Carter model as well as over the model based on the Heligman-Pollard function and ARIMA processes. As the forecast horizon is lengthened, a model based on the Heligman-Pollard function and VECM becomes a source of increased accuracy. In contrast to the Lee-Carter model, the Heligman-Pollard function uses a smaller set of parameters as it enforces a particular age structure, and thus, projections can less easily be unrealistic. Furthermore, VECM take into account dependencies across age-groups. The resulting forecasts are more reasonable and allow a more realistic quantification of risk. Although, our results should be interpreted with caution, as longer time horizons are needed in order to confirm our assumption of improvement in accuracy of a VECM over longer forecast periods. A period of 50 years of observation is known to be quite short for a VECM analysis. As a result, model coefficients are estimated with large standard errors.

In the second part of the paper, VECM are developed for cause of death factors and appear to capture accurately the dynamics of cause-specific mortality. Our analysis shows that long-run equilibrium relationships exist between the five main causes of death, improving our understanding of the dependence between competing risks. Thus, the usually made assumption of independence between causes is erroneous and may lead to inaccurate analyses. It is important to incorporate these common trends in models used for risk management, especially for risk diversification as groups of countries have similar experience. New models incorporating this information need still to be developed.

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Appendices

A Percentages of Deaths by Cause, Females

Table 12: Sample years of the percentages of deaths by cause, females

Country	Cause of death	1955	1970	1985	2000
USA (1950 - 2005)	Circulatory system	40.70%	55.92%	50.10%	41.09%
	Cancer	17.89%	18.17%	22.08%	22.35%
	Respiratory system	3.78%	4.76%	7.39%	9.82%
	External causes	5.23%	5.57%	4.17%	3.80%
	I&P	1.38%	0.82%	1.34%	2.22%
	Total	68.99%	85.23%	85.09%	79.28%
Australia (1950 - 2003)	Circulatory system	36.79%	57.67%	52.14%	42.21%
	Cancer	16.10%	16.29%	22.16%	25.76%
	Respiratory system	5.00%	5.87%	6.35%	8.10%
	External causes	4.72%	5.58%	4.33%	4.19%
	I&P	1.22%	0.78%	0.59%	1.27%
	Total	63.83%	86.18%	85.56%	81.54%
Switzerland (1951 - 2005)	Circulatory system	34.79%	49.29%	49.38%	43.25%
	Cancer	18.65%	20.93%	25.29%	22.40%
	Respiratory system	5.65%	5.60%	4.66%	6.87%
	External causes	4.81%	6.21%	6.92%	4.26%
	I&P	2.61%	1.10%	0.55%	1.17%
	Total	66.51%	83.13%	86.81%	77.95%
Japan (1950 - 2006)	Circulatory system	10.40%	43.34%	43.63%	35.27%
	Cancer	11.90%	17.10%	23.21%	27.67%
	Respiratory system	6.64%	7.62%	8.73%	13.53%
	External causes	5.41%	5.70%	5.08%	5.62%
	I&P	8.95%	2.83%	1.32%	2.05%
	Total	43.29%	76.58%	81.97%	84.15%
Singapore (1963 - 2006)	Circulatory system	NA	27.48%	37.23%	40.08%
	Cancer	NA	14.60%	20.81%	25.99%
	Respiratory system	NA	13.53%	15.68%	15.64%
	External causes	NA	4.76%	5.09%	3.61%
	I&P	NA	4.85%	2.49%	1.77%
	Total	0.00%	65.23%	81.32%	87.10%

Table 12: Sample years of the percentages of deaths by cause, females - continue

Country	Cause of death	1955	1970	1985	2000
Italy (1951 - 2002)	Circulatory system	30.36%	49.87%	50.49%	47.60%
	Cancer	14.85%	17.75%	21.47%	24.40%
	Respiratory system	8.43%	8.87%	5.43%	5.68%
	External causes	2.30%	3.52%	4.04%	3.69%
	I&P	3.09%	1.47%	0.35%	0.65%
	Total	59.02%	81.47%	81.80%	82.02%
Norway (1951 - 2005)	Circulatory system	29.05%	50.87%	48.48%	42.66%
	Cancer	19.81%	19.71%	21.62%	22.68%
	Respiratory system	6.59%	10.25%	10.81%	10.44%
	External causes	3.96%	4.40%	4.90%	4.22%
	I&P	1.75%	0.78%	0.73%	1.33%
	Total	61.14%	86.01%	86.54%	81.33%
Sweden (1951 - 2005)	Circulatory system	36.15%	53.48%	55.06%	46.86%
	Cancer	19.08%	22.05%	21.56%	22.17%
	Respiratory system	5.50%	6.02%	7.98%	7.11%
	External causes	3.57%	5.10%	4.06%	3.28%
	I&P	1.28%	0.84%	0.76%	1.08%
	Total	65.58%	87.48%	89.41%	80.48%
United Kingdom (1950 - 2006)*	Circulatory system	37.92%	53.44%	49.58%	39.85%
	Cancer	17.42%	19.03%	22.40%	24.23%
	Respiratory system	8.82%	12.96%	10.43%	12.97%
	External causes	3.40%	3.58%	2.70%	2.41%
	I&P	1.34%	0.53%	0.39%	0.82%
	Total	68.89%	89.54%	85.51%	80.28%

I&P = Infectious and parasitic diseases.
The years in brackets under the country name represent the period under observation.
The table should be read as follows: In 2000, in the United States, 41.09% of the deaths were because of the diseases of the circulatory system. The five main causes of death caused 79.28% of the deaths.

* No data are available for United Kingdom in 2000. The percentages of 2001 are presented instead.

B International Classification of Diseases

Table 13: Coding system

Cause of death	ICD 7	ICD 8	ICD 9	ICD 10	
				Switzerland	Other countries
Circulatory system	A079-A086	A080-A088	B25-B30	1064	I00-I99
Cancer	A044-A060	A045-A061	B08-B17	1026	C00-D48
Respiratpry system	A087-A097	A089-A096	B31-B32	1072	J00-J99
External causes	A138-A150	A138-A150	B47-B56	1095	V00-Y89
Infectious and parasitic diseases	A001-A043	A001-A044	B01-B07	1001	A00-B99

Table 14: Adoption of new classifications

Country	ICD change	Year
USA	ICD7-8	1968
	ICD8-9	1979
	ICD9-10	1999
Australia	ICD7-8	1968
	ICD8-9	1979
	ICD9-10	1998
Switzerland	ICD7-8	1969
	ICD8-10	1995
Japan	ICD7-8	1968
	ICD8-9	1979
	ICD9-10	1995
Singapore	ICD7-8	1969
	ICD8-9	1979

Country	ICD change	Year
Italy	ICD7-8	1968
	ICD8-9	1979
Norway	ICD7-8	1969
	ICD8-9	1986
	ICD9-10	1996
Sweden	ICD7-8	1969
	ICD8-9	1987
	ICD9-10	1997
United Kingdom	ICD7-8	1968
	ICD8-9	1979
	ICD9-10	2001

The International Classification of Diseases changed three times between 1950 and 2006. The aim of such changes was to take into account progresses in science and technology as well as to refine the categories of the diseases in order to have a more detailed description. With ICD-7, the death numbers were classified in 150 different categories. In ICD-10, 11'468 categories and subcategories exist.

C Heligman-Pollard Function: Demographic Meaning of the Parameters

The Heligman-Pollard function used in this paper is introduced in Section 4.1 as

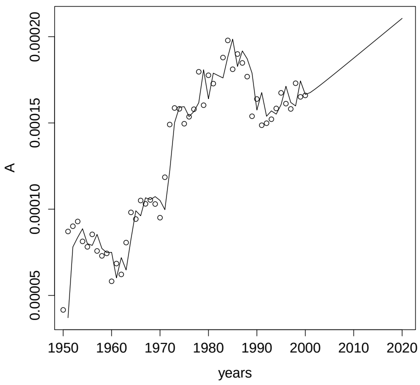
$$q_{x,t} = A_t^{(x+B_t)^{C_t}} + D_t e^{-E_t(\ln(x)-\ln(F_t))^2} + \frac{G_t H_t^x}{1+K_t G_t H_t^x}.$$

The following table explains in some details the demographic meaning of the nine parameters.

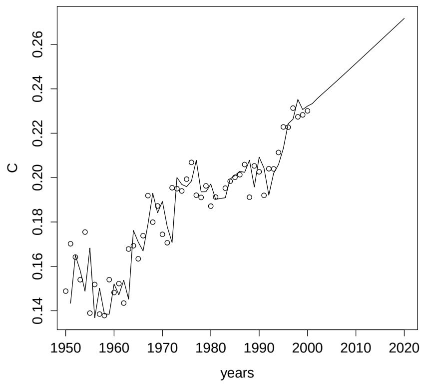
Table 15: Parameters of the Heligman-Pollard function

Age range	**Parameter**	**Demographic meaning**
Childhood	A	Level of infant mortality
	B	Mortality change between age zero and age one (the larger the value, the smaller the change)
	C	Speed at which the mortality rates decline at young ages
Middle ages: Accident hump	D	Intensity of the accident hump
	E	Age range at which the accident hump occurs (varies inversely with the spread of the hump)
	F	Age at which the hump is the highest
Older ages	G	General level of mortality rates
	H	Steepness of the curve
	K	Curvature at the last ages (a positive value indicates that the age pattern of mortality on a semi-logarithmic scale is concave downward; a negative value indicates a concave upward curve)

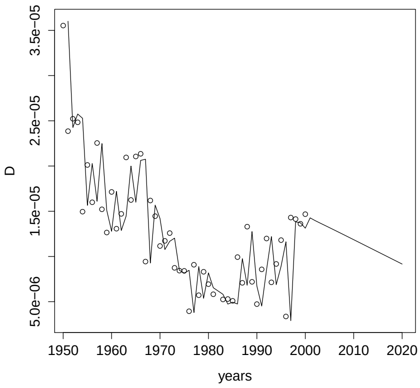
D Trend of the Parameters of the Heligman-Pollard Function



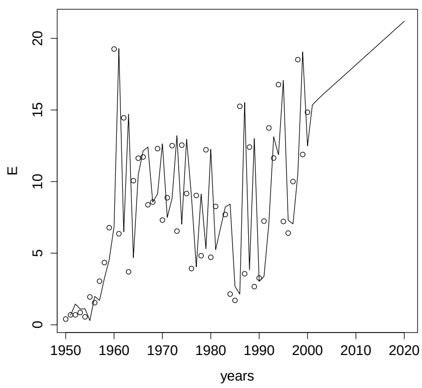
(a) Parameter A



(b) Parameter C

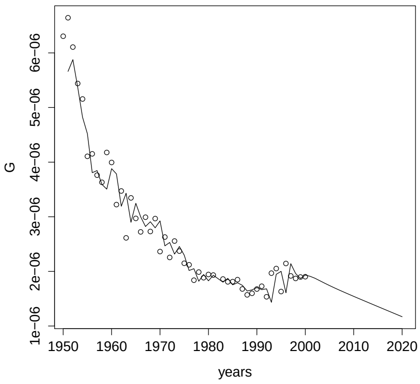


(c) Parameter D

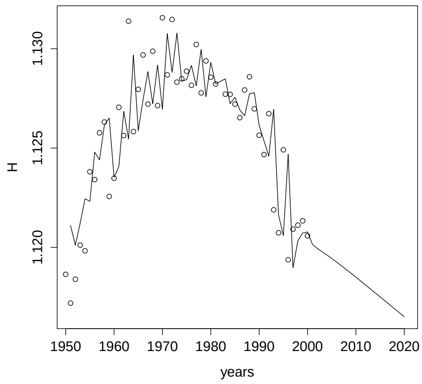


(d) Parameter E

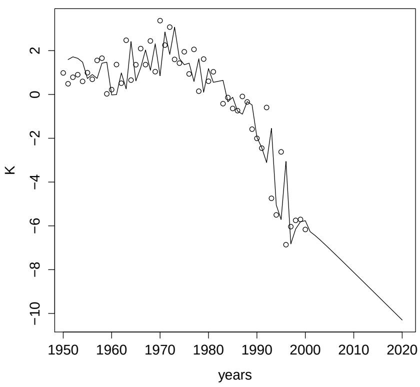
Figure 7: Parameter history and forecasts of the Heligman-Pollard function, Circulatory system, USA, females



(e) Parameter G



(f) Parameter H

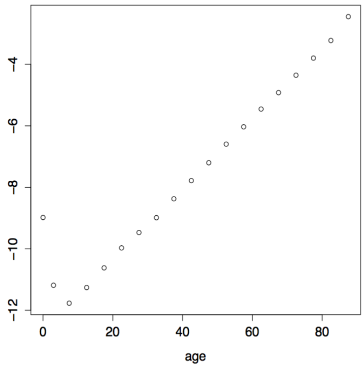


(g) Parameter K

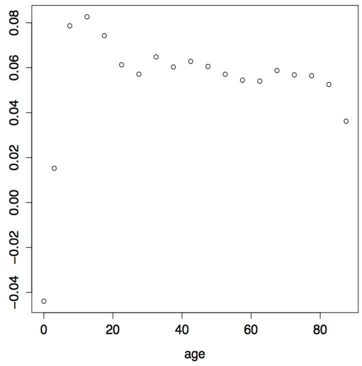
Figure 7: Parameter history and forecasts of the Heligman-Pollard function, Circulatory system, USA, females - continue

The Heligman-Pollard function is fitted on mortality rates over the period 1950–2000. The dots in the graphs represent the resulting values of the parameters of this function. The parameters are then modeled using a VECM, which is depicted by the curves, along with the forecasted values (from 2001 to 2020).

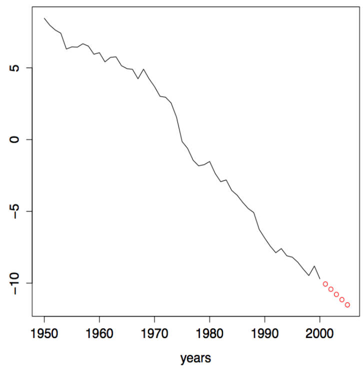
E Parameters of the Lee-Carter Model



(a) Parameter α



(b) Parameter β



(c) Parameter κ

Figure 8: Parameters of the Lee-Carter model, Circulatory system, USA, females

The parameters are estimated by maximum likelihood as described in Section 4.2. Parameter κ is forecasted from 2001 to 2005 (dots) using an ARIMA(0,1,0) model with a drift of -0.36 described in Section 4.2.

F Parameters of the Heligman-Pollard Function: ARIMA Processes

ARIMA$(k, 1, (k-1))$ models are successively fitted on each parameter of the Heligman-Pollard function, increasing k by one, as suggested by Pandit and Wu [2001]. We use the F-criterion in order to decide which model between an ARIMA$(k, 1, (k-1))$ and an ARIMA$((k+1), 1, k)$ is the most adequate. The F-criterion is a test allowing to check for the improvement in the residual sum of squares of the error, ϵ_t. It is a test of the assumption that some of the coefficients in a model are restricted to zero.[6] Once the appropriate ARIMA$(k, 1, (k-1))$ process is found, we check that the confidence interval of their coefficients does not include zero. If some intervals include zero, the F-criterion is applied in order to determine the adequacy of a model without the corresponding coefficients. Finally, we check that the residuals of the fitted model are uncorrelated. The following table describes the most adequate ARIMA models according to this procedure.

[6]For a detailed description, see Pandit and Wu [2001]

Table 16: Fitted ARIMA$(p, 1, q)$ processes, Circulatory system, USA, females

	∇A: ARIMA(2,1,2)			∇C: ARIMA(2,1,1)			∇D: ARIMA(2,1,0)			∇E: ARIMA(3,1,2)		
	Values	CI 95%		Values	CI 95%		Values	CI 95%		Values	CI 95%	
ϕ_1	-0.66	-1.01	-0.31	-1.36	-1.58	-1.13	-0.72	-1.00	-0.44	0.37	0.16	0.59
ϕ_2	-0.55	-0.82	-0.27	-0.58	-0.81	-0.35	-0.36	-0.63	-0.08	-0.24	-0.46	-0.02
ϕ_3										-0.65	-0.86	-0.43
θ_1	0.76	0.57	0.96	1.00	0.87	1.13				-1.27	-1.42	-1.12
θ_2	1.00	0.80	1.20							1.00	0.81	1.19
$\sum \epsilon_t^2$	$6.58 \cdot 10^{-9}$			$4.15 \cdot 10^{-3}$			$7.56 \cdot 10^{-10}$			713.81		

	∇G: ARIMA(2,1,1)			∇H: ARIMA(2,1,1)			∇K: ARIMA(2,1,0)		
	Values	CI 95%		Values	CI 95%		Values	CI 95%	
ϕ_1	0.49	0.21	0.77	-1.48	-1.69	-1.28	-0.71	-0.98	-0.45
ϕ_2	0.48	0.22	0.75	-0.64	-0.85	-0.43	-0.29	-0.55	-0.03
θ_1	-0.84	-1.04	-0.65	1.00	0.88	1.12			
$\sum \epsilon_t^2$	$5.00 \cdot 10^{-12}$			$1.65 \cdot 10^{-4}$			56.71		

All models are identified and estimated using the 50 observations from 1950 to 2000. First differencing transformation is performed on every parameter.

G VECM on Causes of Death across Nine Countries

Tables 17, 18 and 19 give all the information needed in order to represent a VECM. For example, the resulting VECM for males in the United States is

$$\begin{bmatrix} \nabla CircSyst_t \\ \nabla Cancer_t \\ \nabla RespSyst_t \\ \nabla ExtCauses_t \\ \nabla I\&P_t \end{bmatrix}$$

$$= \begin{bmatrix} 0.43874 \\ 0.46001 \\ -0.48234 \\ -0.15794 \\ -4.20466 \end{bmatrix} + \begin{bmatrix} 0.00736 & 0.00606 \\ 0.00700 & 0.00454 \\ 0.00124 & 0.02903 \\ -0.00317 & -0.00497 \\ -0.04774 & 0.02163 \end{bmatrix}$$

$$\times \begin{bmatrix} 1.03933 & -2.34554 & -0.41691 & -6.95797 & -2.15630 \\ -4.37272 & -11.39015 & 8.37977 & 5.60970 & 1.64404 \end{bmatrix}$$

$$\times \begin{bmatrix} CircSyst_{t-1} \\ Cancer_{t-1} \\ RespSyst_{t-1} \\ ExtCauses_{t-1} \\ I\&P_{t-1} \end{bmatrix}. \qquad (20)$$

Table 17: Constants included in a VECM based on the five main causes of death

	Males					Females				
	Circulatory system	Cancer	Respiratory system	External causes	I&P	Circulatory system	Cancer	Respiratory system	External causes	I&P
USA	0.43874	0.46001	-0.48234	-0.15794	-4.20466	0.73665	-0.34753	-2.32401	-1.39652	-7.53256
Australia	-3.89968	-2.26731	-14.24828	-5.56537	-8.19684	-1.08802	-0.42182	-6.84809	-0.51955	-2.34092
Switzerland	-0.65890	-0.95909	4.13417	-2.03313	6.07545	-	-	-	-	-
Japan	-0.58047	-0.39123	0.65844	-0.31950	0.93573	4.43095	-1.99409	18.65621	2.16433	4.17431
Singapore	-	-	-	-	-	-0.18357	-0.16679	0.45846	-0.62157	0.73467
Italy	-0.55202	-0.18801	-0.50883	-0.12029	0.02746	1.06598	-0.06050	5.19679	0.67211	0.39017
Norway	1.63403	-0.13604	-0.62175	0.74013	-6.38061	-2.48457	-3.01543	-9.70387	-1.50241	2.16388
Sweden	-0.12201	-0.20726	-1.45263	1.49949	-7.54633	-1.98343	-0.41957	-8.14290	0.42770	5.82425
UK	-1.32293	-0.41679	-5.06989	0.07603	-0.62117	-0.04302	-0.00512	-0.21213	0.05591	-0.39890

I&P = Infectious and parasitic diseases.
One VECM is fitted by country, following the Johansen's procedure. The variables used in a VECM are the five main causes of death of the country under study. The constants included in a model are described by the line of this table corresponding to the particular country under study. For example, 0.44 is the constant needed for the diseases of the circulatory system for males in the United States, while -4.2 is used for the infectious and parasitic diseases.

Males in Singapore as well as females in Switzerland are not represented in this table as no cointegrated relation is found and so, no VECM is fitted, but a VAR (Appendix H).

Table 18: Cointegrated relations between the five main causes of death

		Males					Females				
		Circulatory system	Cancer	Respiratory system	External causes	I&P	Circulatory system	Cancer	Respiratory system	External causes	I&P
USA	**1st**	1.03933	-2.34554	-0.41691	-6.95797	-2.15630	-2.46855	18.79722	-0.11446	6.42895	-0.23762
	2nd	-4.37272	-11.39015	8.37977	5.60970	1.64404					
Australia	**1st**	-2.56301	-15.06544	-9.82005	11.56377	-3.71314	-11.78058	-6.42785	-8.05339	10.96734	2.66325
	2nd	-1.91375	11.44725	1.61785	-1.11874	0.50319					
	3rd	18.49540	-20.37665	-0.69072	-28.33015	-1.36802					
Switzerland	**1st**	-14.89733	17.51768	-	17.78248	-0.64395	-	-	-	-	-
Japan	**1st**	0.99443	-4.61224	-	-3.33358	1.19568	5.41306	-36.97359	-	5.49919	-0.46654
	2nd						-10.42972	56.45083	-	-1.10891	2.11263
	3rd						-2.41726	-27.85372	-	21.30805	-2.97511
Singapore	**1st**	-	-	-	-	-	2.10568	0.35377	-7.93496	6.45040	-
Italy	**1st**	8.41799	15.32077	-	-18.74788	2.39752	15.32377	11.19038	-13.14499	-4.89043	4.19417
Norway	**1st**	-6.79880	-8.80674	-	4.30630	-2.52123	5.90035	42.09067	-	-3.49216	-2.77631
Sweden	**1st**	12.07646	16.98234	-	-13.20739	3.88387	-4.60714	30.86239	6.80098	-3.54985	-3.97700
UK	**1st**	4.55206	-11.07908	-1.43379	3.57710	-0.75337	-2.70738	2.08495	-	1.31605	-1.53318
	2nd	1.31480	-25.36101	-6.91611	18.38995	-4.22144					

I&P = Infectious and parasitic diseases.
One VECM is fitted by country, following the Johansen's procedure. The variables used in a VECM are the five main causes of death of the country under study. The cointegrated relations included in the model corresponding to a particular country are described in this table. The number of required cointegrated relations are discussed in Section 4.3 and presented in Table 8. For example, a VECM representing female mortality in the United States uses one long-run equilibrium relationship (cointegrated relation), that we may write as

$$-2.47 \times CircSyst_t + 18.80 \times Cancer_t - 0.11 \times RespSyst_t + 6.43 \times ExtCauses_t - 0.24 \times I\&P_t = z_t,$$

where z_t is a stationary variable.

Males in Singapore as well as females in Switzerland are not represented in this table as no cointegrated relation is found and so, no VECM is fitted, but a VAR (Appendix H).

Table 19: Loadings of a VECM based on the five main causes of death

		Males					Females				
		Circulatory system	Cancer	Respiratory system	External causes	I&P	Circulatory system	Cancer	Respiratory system	External causes	I&P
USA	**1st**	0.00736	0.00700	0.00124	-0.00317	-0.04774	-0.00503	0.00230	0.01550	0.00922	0.05008
	2nd	0.00606	0.00454	0.02903	-0.00497	0.02163					
Australia	**1st**	-0.01737	-0.00549	-0.08965	-0.00188	-0.04695	-0.01942	-0.00765	-0.12463	-0.00921	-0.04206
	2nd	0.01969	0.00929	0.02488	0.00867	-0.04378					
	3rd	-0.00149	-0.00393	-0.00527	-0.01955	-0.02153					
Switzerland	**1st**	0.00472	0.00705	-0.03053	0.01489	-0.04505	-	-	-	-	-
Japan	**1st**	-0.01638	-0.01147	0.01947	-0.00910	0.02846	-0.01016	-0.00833	0.04035	-0.00232	0.04410
	2nd						-0.02138	0.00219	-0.04205	-0.00557	0.00866
	3rd						-0.00472	-0.00106	0.00108	0.01873	-0.00274
Singapore		-	-	-	-	-	0.01230	0.01217	-0.03460	0.04579	-0.05760
Italy	**1st**	0.02576	0.00927	0.02326	0.00539	-0.00318	-0.01927	0.00110	-0.09332	-0.01192	-0.00762
Norway	**1st**	0.01965	-0.00165	-0.00750	0.00892	-0.07600	0.01034	0.01262	0.04062	0.00627	-0.00915
Sweden	**1st**	0.00108	0.00196	0.01378	-0.01425	0.07121	0.01366	0.00289	0.05640	-0.00302	-0.04043
UK	**1st**	-0.00866	-0.00895	-0.00027	-0.00463	0.04552	-0.00352	-0.00072	-0.03381	0.01202	-0.06217
	2nd	-0.01264	-0.00185	-0.06068	0.00265	-0.02299					

I&P = Infectious and parasitic diseases.
One VECM is fitted by country, following the Johansen's procedure. The variables used in a VECM are the five main causes of death of the country under study. The impact of each cointegrated relation (described in Table 18) on a cause of death is presented in this table. For example, the diseases of the circulatory system of males in the United States are affected by the first cointegrated relation with a factor of 0.00736, while the second cointegrated relation has an impact of 0.00606.

Males in Singapore as well as females in Switzerland are not represented in this table as no cointegrated relation is found and so, no VECM is fitted, but a VAR (Appendix H).

H Special Cases of Singapore and Switzerland: VAR

Table 20: Autoregressive coefficients as well as the trend used in the VAR associated to males in Singapore

	Δ Circulatory system (t)	Δ Cancer (t)	Δ Respiratory system (t)	Δ External causes (t)	Infectious and parasitic diseases (t)
Δ Circulatory system (t-1)	-0.46283	-0.02341	-0.21527	-0.09648	-0.30539
Δ Cancer (t-1)	0.14898	-0.49717	-0.05213	-0.04751	0.76079
Δ Respiratory system (t-1)	0.13283	0.00244	-0.40142	0.06997	-0.11157
Δ External causes (t-1)	0.13739	0.17410	-0.09101	-0.51364	-0.38760
Infectious and parasitic diseases (t-1)	-0.00562	-0.00500	-0.00946	-0.00362	1.00439
Trend	-0.00262	-0.00212	-0.00392	-0.00228	-0.00153

The VAR is fitted on the first difference of the non-stationary variables, that is on the first difference of the diseases of the circulatory system, cancer, the diseases of the respiratory system and the external causes of death. The infectious and parasitic diseases are stationary and thus, no differentiation is required.
The table should be read as follows: The first difference of mortality due to cancer at time $t-1$ impacts the first difference of mortality due do the diseases of the circulatory system at time t through the coefficient 0.149. The diseases of the circulatory system are affected by the five causes of death as follows

$$\begin{aligned} \nabla CircSyst_t = & - 0.46283 \times \nabla CircSyst_{t-1} + 0.14898 \times \nabla Cancer_{t-1} \\ & + 0.13283 \times \nabla RespSyst_{t-1} + 0.13739 \times \nabla ExtCauses_{t-1} \\ & - 0.00562 \times I\&P_{t-1} - 0.00262 \times t. \end{aligned}$$

Table 21: Autoregressive coefficients, the constant as well as the trend used in the VAR associated to females in Switzerland

	Δ Circulatory system (t)	Cancer (t)	Respiratory system (t)	Δ External causes (t)	Δ Infectious and parasitic diseases (t)
Δ Circulatory system (t-1)	0.14614	0.03040	-0.16896	0.21790	0.30522
Cancer (t-1)	-1.42405	0.16423	-1.10560	-0.60341	-0.12140
Respiratory system (t-1)	-0.09331	-0.02942	0.48664	-0.04918	-0.14489
Δ External causes (t-1)	0.17695	0.17611	0.44943	-0.18828	0.07370
Δ Infectious and parasitic diseases (t-1)	0.02529	0.05968	-0.00555	-0.03199	-0.17226
Trend	-0.00817	-0.00435	-0.01076	-0.00384	-0.00043
Constant	-8.96603	-5.08569	-10.01289	-3.86021	-1.78953

The VAR is fitted on the first difference of the non-stationary variables, that is on the first difference of the diseases of the circulatory system, the external causes of death and the infectious and parasitic diseases. Cancer and the diseases of the respiratory system are stationary and thus, no differentiation is required.
The table should be read as follows: Mortality due to cancer at time $t-1$ impacts the first difference of mortality due do the diseases of the circulatory system at time t through the coefficient -1.42. The diseases of the circulatory system are affected by the five causes of death as follows

$$\begin{aligned} \nabla CircSyst_t = & + & 0.14614 \times \nabla CircSyst_{t-1} - 1.42405 \times Cancer_{t-1} \\ & - & 0.09331 \times RespSyst_{t-1} + 0.17695 \times \nabla ExtCauses_{t-1} \\ & + & 0.02529 \times \nabla I\&P_{t-1} - 0.00817 \times t - 8.96603. \end{aligned}$$