The Optimal Product Portfolios for Hedging Longevity Risks and Financial Risks for Life Insurers: A Dynamic Immunization Approach

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Introduction

In the past decade, longevity phenomenon have taken place across human society. Benjamin and Soliman (1993), and McDonald et al. (1998) confirm that unprecedented improvements in population longevity have occurred worldwide. The decrease of mortality rate creates a great risk for insurance company operations. Existing literature proposes many solutions to mitigate the threat of longevity risk for life insurance. We can classify these studies into three categories: capital market solution, industry self-insurance solution and mortality projection improvement. *Capital market solution* include mortality securitization (see, for example, Dowd 2003; Lin and Cox 2005; Blake et al. 2006a, 2006b; Cox et al. 2006), survivor bonds (e.g., Blake and Burrows 2001; Denuit et al. 2007), and survivor swaps (e.g., Dowd et al. 2006). These studies suggest that insurance companies could transfer their exposures to the capital markets which have more funding and participators. Cowley and Cummins (2005) give an overview on the securitizations of life insurance asset and liability. The *industry self-insurance solution* include the nature hedging strategy of Cox and Lin (2007), the duration matching strategy of Wang et al. (2009), and the Conditional Value at Risk approach of Wang et al. (2010). The advantages of industry self-insurance solutions are there may exist a lower transaction cost and insurance companies do not need a liquid market. They can hedge longevity risk by themselves or with other counterparties. Third solution, the *mortality projection improvement*, gives a more accurate estimation of and more realistic assumptions about mortality processes. As Blake et al. (2006b) suggest, these works fall into two areas: continuous-time frameworks (e.g., Milevsky and Promislow 2001; Dahl 2004; Biffis 2005; Schrager 2006) and discrete-time frameworks (e.g., Brouhns et al. 2002; Renshaw and Haberman 2003; Cairns et al. 2006b). The parameter uncertainty and model specification on mortality process also be paid more attention in recent (e.g. Melnikov and Romaniuk, 2006, Koissi et al., 2006 and Wang et al. 2010).

With the launch of new products, such as reverse mortgages(or reverse annuity) and equity-linked annuities, more liability risk and financial risks are involved for life insurance companies. It increases the complexity and challeges to implement the asset liability management strategy. However, some financial risks are not postive corelated (e.g. interest rate and real estate) which may provide diversication or even hedge effect among asset portfolio. In addition, with a hedge final payment received at the end of the policy year, reverse mortgages provide different liability duration from annuity. It may also create a hedge effect for life insurers in terms of cash flows patterns. In this paper, we propose a generalized immunization approach to obtain an product portfolio for hedging longevity and financial risk for life insurance companies. We assume that insurers sell muti products, including life insurance, traditional annutiy, equity-linked annuities and reserve mortgages, etc… Similar to the duration match approach, it is common practice to use delta hedge strategy to mitigate the risks associated with the movements of these financial risk factors. We consider the simultaneous shocks to mortality curve, interest rate curve, and other financial risk factors existing in the product portfolio under consideration. We first adopt the two-factor stochastic mortality model (Cairns et al., 2006b, the CBD model) to construct future mortality processes and corresponded liability distributions. We also simulate interest rate curve by using Cox, Ingersoll, and Ross (1985) stochastic interest model (the CIR model). We demonstrate that the proposed dynamic immunization approach can lead to an optimal product portfolio which can simultaneously reduce longevity risks, interest rate risk, and other financial risks existing in a product portfolio effectively. By means of the stochastic simulation of the movements in mortality curve, interest rate curve and related risk factors, we show that the proposed immunization approach can be served as an effective vehicle to significantly reduce the aggregate risk for life insruance companies.

Proposed dynamic Immunization Approach

Let q, r, and S be the current mortality curve, the interest rates curve, and other financial risk factors, respectively. The current value of the product portfolio with hedging assets is $V(q, r, S)$. Assume that q_s, r_s, and S_s are typical shocks of the mortality curve, the interest rates curve, and other financial risk factors, respectively. These typical shocks can be estimated from historical data. For example, it is customary to assume r_s is a parallel shift (Willner 1996). However, other types of shocks are possible (Willner 1996; Golub and Tilman 1997). In our model assumptions, these typical shocks are model parameters. Please note that q, r, S, q_s, r_s, and S_s are vectors. Now assume that the simultaneous shocks to mortality curve, interest rate curve, and other financial risk factors are $\Delta q \times q_s$, $\Delta r \times r_s$, and $\Delta S \times S_s$. Here, Δq, Δr, and ΔS represent the magnitudes of the movements in the mortality curve, the interest rates curve, and other financial risk factors, respectively.

Denote P the product portfolio with hedging assets. Therefore, after the shocks, the value of P becomes $V(q+\Delta q \times q_s, r+\Delta r \times r_s, S+\Delta S \times S_s)$. Through the Taylor's expansion, we have

$$V(q+\Delta q q_{s'}, \quad r+\Delta r r_{s'}, \quad S+\Delta S S_{s'}) - V(q,r,S)$$
$$\approx \left(\frac{\partial V}{\partial \Delta q}\Delta q + \frac{\partial V}{\partial \Delta r}\Delta r + \frac{\partial V}{\partial \Delta S}\Delta S\right) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial \Delta q^2}(\Delta q)^2 + \frac{\partial^2}{\partial \Delta r^2}(\Delta r)^2 + \frac{\partial^2 V}{\partial \Delta S^2}(\Delta S)^2\right)$$
$$+\left(\frac{\partial^2 V}{\partial \Delta q \partial \Delta r}\Delta q\,\Delta r + \frac{\partial^2 V}{\partial \Delta r \partial \Delta S}\Delta r \Delta S + \frac{\partial^2 V}{\partial \Delta q \partial \Delta S}\Delta q\,\Delta S\right) \qquad (1)$$

Assume that P has n products and hedging assets. Let V_k denote the value of k-th product or hedging asset of P and w_k denote the number of units sold (if it is a product) or the number of shares purchased (if it is a hedging asset). Then

$$\frac{\partial V}{\partial \Delta q} = \sum_{k=1}^{n} \omega_k \frac{\partial V_k}{\partial \Delta q}, \quad \frac{\partial V}{\partial \Delta r} = \sum_{k=1}^{n} \omega_k \frac{\partial V_k}{\partial \Delta r}, \quad \frac{\partial V}{\partial \Delta S} = \sum_{k=1}^{n} \omega_k \frac{\partial V_k}{\partial \Delta S}$$

$$\frac{\partial^2 V}{\partial \Delta q^2} = \sum_{k=1}^{n} \omega_k \frac{\partial^2 V_k}{\partial \Delta r^2}, \quad \frac{\partial^2 V}{\partial \Delta r^2} = \sum_{k=1}^{n} \omega_k \frac{\partial^2 V_k}{\partial \Delta r^2}, \quad \frac{\partial^2 V}{\partial \Delta S^2} = \sum_{k=1}^{n} \omega_k \frac{\partial^2 V_k}{\partial \Delta S^2}$$

$$\frac{\partial^2 V}{\partial \Delta q \partial \Delta r} = \sum_{k=1}^{n} \omega_k \frac{\partial^2 V_k}{\partial \Delta q \partial \Delta r}, \quad \frac{\partial^2 V}{\partial \Delta r \partial \Delta S} = \sum_{k=1}^{n} \omega_k \frac{\partial^2 V_k}{\partial \Delta r \partial \Delta S},$$

$$\frac{\partial^2 V}{\partial \Delta q \partial \Delta S} = \sum_{k=1}^{n} \omega_k \frac{\partial^2 V_k}{\partial \Delta q \partial \Delta S} \qquad (2)$$

With suitable choice of w_k, we can make

$$\frac{\partial V}{\partial \Delta q}, \frac{\partial V}{\partial \Delta r}, \frac{\partial V}{\partial \Delta S} \qquad (3)$$

all equal 0 (this is usually achievable when $n \geq 3$); or

$$\frac{\partial V}{\partial \Delta q}, \frac{\partial V}{\partial \Delta r}, \frac{\partial V}{\partial \Delta S}, \frac{\partial^2}{\partial \Delta q^2}, \frac{\partial^2}{\partial \Delta r^2}, \frac{\partial^2}{\partial \Delta S^2}, \frac{\partial^2 V}{\partial \Delta q \partial \Delta r}, \frac{\partial^2 V}{\partial \Delta r \partial \Delta S}, \frac{\partial^2 V}{\partial \Delta q \partial \Delta S} \qquad (4)$$

all equal 0 (this is usually achievable when $n \geq 9$).

If (3) holds, we have immunized product portfolio against small movements in the mortality curve, the interest rates curve, and other financial risk factors simultaneously. In this case, *P* is similar to zero duration or delta neutral portfolios.

If (4) holds, we have immunized product portfolio against larger movements in the mortality curve, the interest rates curve, and other financial risk factors simultaneously. In this case, *P* is similar to zero duration and zero convexity portfolios or delta neutral and gamma neutral portfolios.

We select a variety of product portfolios to test the effectiveness of the proposed immunization approach,. By means of the stochastic simulation of the movements in mortality curve, interest rate curve and related risk factors, we show that the proposed immunization approach can simultaneously reduce longevity risks, interest rate risk, and other financial risks existing in a product portfolio effectively.

The Two-Factor Stochastic Mortality Model

Several stochastic models proposed in the literature attempt to capture the processes of mortality rate. We choose the two-factor mortality model (the CBD model) as the underlying mortality process for two reasons. First, the CBD model characterize not only a cohort effect but also a quadratic age effect. The two factors $A_1(t)$ and $A_2(t)$ in the CBD model represent all age-general improvements in mortality over time and different improvement for different age groups. These two factors reflect the "trend effect" and "age effect". Thus, the analysis will be economically or *biologically* meaningful when we consider the parameter changes of these factors over time. Second, the CBD model is a discrete time model and can be implemented in practice more convenient. We offer a brief description of the two-factor model; for more detailed discussions, see Cairns et al. (2006b).

Let $q_{t,x}$ be the realized mortality rate for age x insured from time t to $t+1$. Assume the mortality curve has a logistic functional form:

$$q_{t,x} = 1 - p = \frac{e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}{1+e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}. \qquad (5)$$

The two stochastic trends $A_1(t+1)$ and $A_2(t+1)$, follow a random walk process with drift parameter μ and diffusion parameter C :

$$A(t+1) = A(t) + \mu + CZ(t+1), \qquad (6)$$

where $A(t+1) = [A_1(t+1),\ A_2(t+1)]^?$, $\mu = [\mu_1,\ \mu_2]^?$ is a 2 × 1 constant parameter vector. C is a 2 × 2 constant upper-triangular Cholesky square root matrix of the covariance matrix $V = CC^?$ and $Z(t)$ is a two-dimensional standard normal random variable. A_1 and A_2 were estimated using least squares from $\log(q_y / p_y) = A_1 + A_2 \cdot y + \varepsilon$ for all t . To include the uncertainty of μ and C , Cairns et al. (2006b) invoke a normal-inverse-Wishart distribution from a non-informative prior distribution perspective:

$$V^{-1} \big| D \sim Wishart(n-1,\ n^{-1}\hat{V}^{-1})$$

$$\mu^{-1} \big| V, D \sim MVN(\hat{\mu},\ n^{-1}V),$$

where $D(t) = A(t) - A(t-1),$ (7)

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} D(t),$$

and $$\hat{V} = \frac{1}{n}\sum_{t=1}^{n}\big(D(t) - \hat{\mu}\big)\big(D(t) - \hat{\mu}\big)^?.$$

Then we can generate $A(t)$ from equation (6) with the parameters μ and C from equation (7). Then we get $q_{t,x}$ as equation (5) suggests.

The CIR Interest Rate model

If the interest rate follows the stochastic process suggested by Cox, Ingersoll, and Ross (1985), then the interest rate path can be expressed as

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\,dz$$

where a , b , and σ are constants and dz follows a standard Brownian motion. The drift rate of the interest rate under above model is $a(b - r_t)$. The standard deviation of the interest rate is $\sigma\sqrt{r_t}$. Cox, Ingersoll, and Ross (1985) solved equation (CIR) and showed that

$$P_t = \alpha_t e^{-\beta_t r}$$

where P_t is the current price of a one-dollar zero-coupon bond of periods and

$$\alpha_t = \left[\frac{2\sqrt{a^2 + 2\sigma^2}\, e^{\frac{t}{2}(a+\sqrt{a^2+2\sigma^2})}}{(\sqrt{a^2 + 2\sigma^2} + a^2)(e^{t\sqrt{a^2+2\sigma^2}} - 1) + 2\sqrt{a^2 + 2\sigma^2}} \right]^{2ab/\sigma^2},$$

$$\beta_t = \frac{2(e^{ta+\sqrt{a^2+2\sigma^2}} - 1)}{(\sqrt{a^2 + 2\sigma^2} + a^2)(e^{t\sqrt{a^2+2\sigma^2}} - 1) + 2\sqrt{a^2 + 2\sigma^2}}.$$

The assumption of the drift rate in Vasicek (1977) and Cox, Ingersoll, and Ross (1985) is the same, whereas the assumption of interest rate variation in these two models is different. Vasicek (1977) assumed that the variation in the interest rate in each period is constant, whereas Cox, Ingersoll, and Ross (1985) assumed that the variation in the interest rate in each period is proportional to the square root of the interest rate in each period. The model risk between Vasicek (1977) and Cox, Ingersoll, and Ross (1985) results from the difference of coefficients α and β rather the difference of function forms. Furthermore, β in equations (CIR beta) is independent of the level of long-term interest rate b. However, under Vasicek's model (1977), b is independent of variance σ; whereas under Cox, Ingersoll, and Ross' model (1985), b depends on variance σ.

Conclusions

To hedge the interest rate risk in insures' liability, asset-liability managers commonly adopt the classical immunization strategy. A similar hedging approach for longevity risk, by matching the mortality duration of life insurance and annuity, has been proposed by Wang et al. (2009 and 2010). With the launch of new innovative products (e.g., reverse mortgages and equity-linked annuities), more financial risks are involved for life insurance companies. with a hedge final payment received at the end of the policy year, reverse mortgages provide different liability duration from annuity. Using the diversication effect among financial ssset and liability duration patterns, we propose a generalized immunization approach to obtain an product portfolio for hedging longevity and financial risk for life insurance companies. We demonstrate that our proposed approach can lead to an optimal product portfolio which can simultaneously reduce longevity risks, interest rate risk, and other financial risks existing in a product portfolio effectively. By means of the historical simulation of the movements in mortality curve, interest rate curve and related risk factors, we show that the proposed immunization approach can be served as an effective vehicle to significantly reduce the aggregate risk for life insruance companies.

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